Minimality of group and monoid presentations

by

Ahmet Sinan Çevik

A thesis submitted to
the Faculty of Science
at the University of Glasgow
for the degree of
Doctor of Philosophy

January, 1998

Contents

STATEMENT

Chapter 1 covers basic material concerning group presentations, monoid presentations and some related topics with them. Most of these are standard and are taken from [49], [50] and [51].

Chapters 2-5 are my own work, with the exception Section 4.3.4, as well as the other material indicated within the text.

ACKNOWLEDGEMENTS

I would like to express my deepest thanks to my supervisor, Prof. S. J. Pride, FRSE, for his inspiration, guidance, and encouragement during the research and preparation of this work.

I would like to acknowledge "Higher Educational Council of Turkey" and "The Republic of Turkey, University of Balıkesir" for their financial support, as well as the Department of Mathematics, University of Glasgow for providing funding to attend conferences in Edinburgh and Bath.

Also, I would like to thank all my friends for all assistance and support.

Finally, special thanks to my family; my wife, my daughter and my son, whose love, support and understanding have made this thesis possible.

ABSTRACT

In Chapter 1 of this thesis we review existing theory concerning group and monoid presentations, and the concept of pictures over these. We also recall aspherical, combinatorial aspherical, n-Cockcroft ($n \in \mathbb{Z}^+$), efficient and inefficient presentations. Minimality is the final concept introduced in this chapter: we present an important theorem, due to Lustig in the case of groups and to Pride for monoids.

In Chapter 2 we prove necessary and sufficient conditions for the presentation of the central extension to be p-Cockcroft (p a prime or 0). The starting point of this result is the joint paper of Baik-Harlander-Pride. We end the chapter by giving some examples.

In Chapter 3 we prove a theorem on the efficiency of standard wreath products of two finite groups. We also present some applications of the theorem and end by giving examples.

Chapter 4 sees discussion on the semi-direct product of any two monoids. In particular we prove necessary and sufficient conditions for the standard presentation of the semi-direct product of any two monoids to be p-Cockcroft (p a prime or 0). We end by giving some applications of this theorem to the direct product of two monoids and the semi-direct product of two finite cyclic monoids.

We begin Chapter 5 with an application of the main theorem of Chapter 4, namely we give necessary and sufficient conditions for a presentation of the semi-direct product of a one-relator monoid by an infinite cyclic monoid to be p-Cockcroft (p a prime or 0), and give some examples of this. Following this we present the main theorem of this chapter, which is sufficient conditions for the presentation of a semi-direct product of a one-relator monoid by an infinite cyclic monoid to be minimal but inefficient. We end by giving some examples.

NOTATION

	Let G and H be groups.
$G \times H$	the direct product
$G \oplus H$	the direct sum (where G, H are abelian)
$G \otimes H$	the tensor product (where G, H are abelian)
$G \rtimes_\theta H$	the semi-direct product of G by H with H-action θ
$G \wr H$	the standard wreath product of G by H
G/H	the quotient group of G by a normal subgroup H
$G \cong H$	G is isomorphic to H
G'	the derived group (commutator subgroup) of G
G^{ab}, $H_1(G)$	the first homology group of G
$H_2(G)$	the second homology group of G (= Schur multiplier)
$Aut(G)$	the group of all automorphisms of G (see note page x)
$[a,b]$	the commutator of a and b ($= aba^{-1}b^{-1}$, $a, b \in G$)
$\mathbb{Z}_n$	the finite cyclic group of order n
$t(A)$	Let A be a non-trivial finite abelian group. Then A can be uniquely decomposed [54] into a direct sum of cyclic groups, that is, $A = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_n}$, where $m_1 > 1$ and $m_i \mid m_{i+1}$ for all $i = 1, 2, \cdots, n-1$. Then $t(A)$ is m_1 (the first torsion number). If A is trivial then $t(A) = 0$
$\mathbb{Z}G$	the integral group ring
$\mathbb{Z}^n$	the free abelian group of rank n
$\mathcal{P} = \langle \mathbf{x} \ ; \ \mathbf{r} \rangle$	group presentation
$F(\mathbf{x})$	the free group generated by $\mathbf{x}$
$G(\mathcal{P})$	group defined by $\mathcal{P}$
$[W]$	free equivalence class containing the word W
$\overline{W}$	the element of $G(\mathcal{P})$ represented by W
$L(W)$	length of W
$L_x(W)$	lenght of W with respect to x

$\exp_x(W)$	the exponent sum of x in W
$\sim$	freely equivalent
$\sim_{\mathcal{P}}$	equivalent (relative to $\mathcal{P}$)
$\chi(\mathcal{P})$	Euler characteristic of $\mathcal{P}$ $(= 1 - \lvert\mathbf{x}\rvert + \lvert\mathbf{r}\rvert)$
$\chi(G)$	Euler characteristic of G
$\delta(G)$	$= 1 - rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G))$
$rk_{\mathbb{Z}}(\ )$	the $\mathbb{Z}$-rank of the torsion free part
$d(\ )$	the minimal number of generators
$def(\mathcal{P})$	deficiency of $\mathcal{P}$
$def(G)$	deficiency of group G
$(T1)^{\pm 1}$, $(T2)^{\pm 1}$	Tietze transformations
$\mathbb{P}$	a picture over $\mathcal{P}$
$\partial\mathbb{P}$	the boundary of $\mathbb{P}$
$W(\mathbb{P})$	the boundary label of $\mathbb{P}$
$< \mathbb{P} >$	the equivalence class containing $\mathbb{P}$
$\pi_2(\mathcal{P})$	the second homotopy module
$\exp_R(\mathbb{P})$	exponent sum of R in $\mathbb{P}$
Δ	disc in the picture $\mathcal{P}$
$\partial\Delta$	boundary of Δ
γ	a transverse path
$W(\gamma)$	the label on γ
$\underline{\gamma}$	a spray
$I_2(\mathcal{P})$	the second Fox ideal over $\mathcal{P}$
$\mathbf{X}$	set of generating pictures of $\pi_2(\mathcal{P})$
	Let M and K be monoids.
$M \rtimes_\theta K$	the semi-direct product of M by K with K-action θ
$M \cong K$	M is isomorphic to K
$End(M)$	the monoid of all endomorphisms of M (see note on page x)

$\mathbb{Z}^{+n}$	the free abelian monoid of rank n
$\mathcal{P} = [\mathbf{y}\ ;\ \mathbf{s}]$	monoid presentation
$\hat{F}(\mathbf{y})$	the free monoid generated by $\mathbf{y}$
$M(\mathcal{P})$	monoid defined by $\mathcal{P}$
W	a positive word on $\mathbf{y}$
$[W]$	free equivalence class containing W
$\overline{W}$	the element of $M(\mathcal{P})$ represented by W
$L(W)$	length of W
$L_y(W)$	length of W with respect to y
$\sim_{\mathcal{P}}$	equivalent (relative to $\mathcal{P}$)
$\chi(\mathcal{P})$	Euler characteristic of $\mathcal{P}$ $(= 1 - \lvert\mathbf{y}\rvert + \lvert\mathbf{s}\rvert)$
$\chi(M)$	Euler characteristic of the monoid M
$\delta(M)$	$= 1 - rk_{\mathbb{Z}}(H_1(M)) + d(H_2(M))$
$rk_{\mathbb{Z}}(\ )$	the $\mathbb{Z}$-rank of the torsion free part
$d(\ )$	the minimal number of generators
$def(\mathcal{P})$	deficiency of $\mathcal{P}$
$def(M)$	deficiency of monoid M
$\dfrac{\partial}{\partial y}$	the Fox derivation for a fixed $y \in \mathbf{y}$
$\mathbb{A}$	an atomic monoid picture
$\mathbb{P}$	a path in $\mathcal{D}(\mathcal{P})$, that is, a picture over $\mathcal{P}$
$\exp_S(\mathbb{P})$	the exponent sum S in $\mathbb{P}$
$I_2^{(l)}(\mathcal{P})$	the right second Fox ideal over $\mathcal{P}$
$I_2^{(r)}(\mathcal{P})$	the left second Fox ideal over $\mathcal{P}$
$\mathcal{D}(\mathcal{P})$	Squier complex
$\mathbf{Y}$	is a trivialiser of $\mathcal{D}(\mathcal{P})$

$\Gamma = (V, E)$ a graph:

V	vertex set
E	edge set
$\iota(e)$	inital vertex of edge e

$\tau(e)$	terminal vertex of edge e
$^{-1}$	inverse function

In set theory:

$A \cup B$	the union of the sets A and B
$A \subseteq B$	A is a subset of B
$\lvert A \rvert$	the cardinality of A

Let $\mathbb{Z}$ and $\mathbb{Z}^{+}$ be the sets of all integer and positive integer numbers.

For any $a, b \in \mathbb{Z}$,

$hcf(a, b)$	highest common factor of a and b

Throughout this thesis, all maps will be written on the left, except when we work with the monoid $End(\ )$ and the group $Aut(\ )$ then we will write maps on the right.

Chapter 1

Preliminaries

1.1 Words

Let $\mathbf{x}$ be a non-empty set. We define $\mathbf{x}^{-1}$ to be a set in one-to-one correspondence with $\mathbf{x}$, $x \leftrightarrow x^{-1}$ ($x \in \mathbf{x}$), and let $\mathbf{x}^{\pm 1} = \mathbf{x} \cup \mathbf{x}^{-1}$. The elements of $\mathbf{x}^{\pm 1}$ are called *letters*. Then, a *word* W (on $\mathbf{x}$) is an expression

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, \tag{1.1}$$

where $n \geqslant 0$, $x_i \in \mathbf{x}$, $\varepsilon_i = \pm 1$ and $1 \leqslant i \leqslant n$. The *initial* letter of W is $\iota(W) = x^{\varepsilon_1}$ and the *terminal* letter of W is $\tau(W) = x^{\varepsilon_n}$. If $n = 0$ then W is the *empty word*, which we denote by 1. We say W is a *positive* word on $\mathbf{x}$ if either $n = 0$ or $n > 0$ and $\varepsilon_i = +1$, $1 \leqslant i \leqslant n$. The *inverse* of W, denoted W^{-1}, is the word

$$x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \cdots x_1^{-\varepsilon_1}.$$

Let W be a word as in (1.1). The *length* of W, denoted by $L(W)$, is the number of the letters involved in W. The length of W with respect to x, denoted by $L_x(W)$, is $\sum_{x_i = x} |\varepsilon_i|$. Also, the exponent sum of x in W, denoted by $\exp_x(W)$, is $\sum_{x_i = x} \varepsilon_i$. If W is empty word then $L_x(W) = 0$ and $\exp_x(W) = 0$. Note that if W is a positive word then $L_x(W) = \exp_x(W)$.

Let W, U be two words on $\mathbf{x}$. The product of W and U, denoted WU, is the *juxtaposition* of W followed by U. By this binary operation, the set $\hat{F}(\mathbf{x})$ of all positive words on $\mathbf{x}$ then is a monoid with identity 1 called the *free monoid* on $\mathbf{x}$.

Two words W, W' on $\mathbf{x}$ are *freely equal*, denoted $W \sim W'$, if one can be obtained from the other by a finite number of applications of the following operations.

(1) : Deletion of a pair of inverse letters $x^{\varepsilon}x^{-\varepsilon}$, $\varepsilon = \pm 1$.

$(1)^{-1}$: Insertion of a pair of inverse letters $x^{\varepsilon}x^{-\varepsilon}$, $\varepsilon = \pm 1$.

We denote the free equivalence class containing W by $[W]$. Let $F(\mathbf{x})$ be the set of all free equivalence classes of words on $\mathbf{x}$. A multiplication can be defined on $F(\mathbf{x})$ by $[W][U] = [WU]$, and one can check that this is well-defined. By this multiplication, $F(\mathbf{x})$ is then a group, the *free group* on $\mathbf{x}$ (see [35, Chapter 1]). We note that sometimes we may simply write W for the free equivalence class $[W]$ for any word W on $\mathbf{x}$, if it does not cause any confusion.

If $W' = UWV$ (U, W, V are words on $\mathbf{x}$) then W is a *subword* of W'. We say that a word on $\mathbf{x}$ is *reduced* if it does not contain any subwords $x^{\varepsilon}x^{-\varepsilon}$ ($x \in \mathbf{x}$, $\varepsilon = \pm 1$). Moreover, $x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n}$ ($n \geqslant 0$, $x_i \in \mathbf{x}$, $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant n$) is *cyclically reduced* if it is reduced and $x_1^{\varepsilon_1} \neq x_n^{-\varepsilon_n}$.

The proof of the following theorem can be found in [18].

Theorem 1.1.1 (Normal Form Theorem) *There is exactly one reduced word in each equivalence class.*

1.2 Group presentations

A *group presentation*

$$\mathcal{P} = \langle \mathbf{x} \; ; \; \mathbf{r} \rangle \tag{1.2}$$

is a pair, where $\mathbf{x}$ is a set (the *generating symbols*) and $\mathbf{r}$ is a set of non-empty, cyclically reduced words on $\mathbf{x}$ (the *relators*). We say that $\mathcal{P}$ is finite if $\mathbf{x}$ and $\mathbf{r}$ are both finite.

We should remark that we will use angular brackets $\langle \cdots \rangle$ to denote a group presentation; square brackets $[\cdots]$ to denote a monoid presentation (see Section 1.3*).*

Throughout this thesis, all group presentations will be assumed to be finite unless stated otherwise.

In order to define a group associated with $\mathcal{P}$, we intoduce the following elementary operations (in addition to the operations (1) and $(1)^{-1}$ above) on words on $\mathbf{x}$. Let W be a word on $\mathbf{x}$.

(2) : If W contains a subword R^ε ($R \in \mathbf{r}$, $\varepsilon = \pm 1$) then delete it.

$(2)^{-1}$: Insert R^ε ($R \in \mathbf{r}$, $\varepsilon = \pm 1$) at any position in W.

Two words W_1, W_2 on $\mathbf{x}$ are *equivalent (relative to $\mathcal{P}$)*, denoted $W_1 \sim_{\mathcal{P}} W_2$, if there is a finite chain of elementary operations of types $(1)^{\pm 1}$, $(2)^{\pm 1}$ leading from W_1 to W_2. Now $\sim_{\mathcal{P}}$ is an equivalence relation on the set of all words on $\mathbf{x}$. Let $[W]_{\mathcal{P}}$ (or simply $[W]$) denote the equivalence class containing W. A multiplication can be defined on equivalence classes by $[W_1]_{\mathcal{P}}.[W_2]_{\mathcal{P}} = [W_1W_2]_{\mathcal{P}}$, and this multiplication is easily checked to be well defined. The set of all equivalence classes together with this multiplication form a group, the *group defined by* $\mathcal{P}$, denoted $G(\mathcal{P})$. The identity in $G(\mathcal{P})$ is $[1]_{\mathcal{P}}$.

A group G is said to be *presented* (or defined) by $\mathcal{P}$ if $G \cong G(\mathcal{P})$.

Let N be the *normal closure* of $\{[R] : R \in \mathbf{r}\}$ in $F(\mathbf{x})$. The proof of the following lemma can be found in [35, Proposition 4].

Lemma 1.2.1

$$G(\mathcal{P}) \cong F(\mathbf{x})/N.$$

We will denote the element $[W]N$ of $F(\mathbf{x})/N$ ($\cong G(\mathcal{P})$) by $\overline{W}$.

1.2.1 Tietze transformations

Let $\mathcal{P} = \langle \mathbf{x} \,;\, \mathbf{r} \rangle$ be a group presentation. We define elementary *Tietze transformations* on $\mathcal{P}$ as follows.

$(T1)$ If $\mathbf{s}$ is a finite set of words on $\mathbf{x}$ and if each $S \in \mathbf{s}$ is a consequence of $\mathbf{r}$ (that is, $[S]$ belongs to the normal closure of $\{[R] : R \in \mathbf{r}\}$), then replace $\mathcal{P}$ by

$$\langle \mathbf{x} \,;\, \mathbf{r}, \mathbf{s} \rangle \,.$$

($T2$) If $\mathbf{t}$ is a finite set of symbols disjoint from $\mathbf{x}$, and U_t ($t \in \mathbf{t}$) is a word on $\mathbf{x}$, then replace $\mathcal{P}$ by

$$\left\langle \mathbf{x}, \mathbf{t} \,;\, \mathbf{r}, t^{-1}U_t \,(t \in \mathbf{t}) \right\rangle .$$

The proof of the following theorem can be found in [47].

Theorem 1.2.2 (Tietze Theorem) *Two presentation $\mathcal{P}_1$ and $\mathcal{P}_2$ define the same group if and only if one can be transformed into the other by a finite number of operations* $(T1)$, $(T1)^{-1}$, $(T2)$, $(T2)^{-1}$.

1.2.2 Pictures over group presentations

The material in this section may also be found in [11] and [49].

Let $\mathcal{P} = \langle \mathbf{x} \,;\, \mathbf{r} \rangle$ be a group presentation. A *picture* $\mathbb{P}$ over $\mathcal{P}$ is a geometric configuration consisting of the following:

(1) A *disc* D^2 with *basepoint* O on the *boundary* ∂D^2 of D^2.

(2) Disjoint *discs* $\Delta_1, \Delta_2, \cdots, \Delta_n$ in the interior of D^2. Each Δ_i has a basepoint O_i on the boundary $\partial \Delta_i$ of Δ_i.

(3) A finite number of disjoint arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ where each arc lies in the closure of $D^2 - \bigcup_{i=1}^{n} \Delta_i$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup \partial \Delta_1 \cup \Delta_2 \cup \cdots \cup \partial \Delta_n$, or is a simple non-closed curve which joins two points of $\partial D^2 \cup \partial \Delta_1 \cup \Delta_2 \cup \cdots \cup \partial \Delta_n$, neither point being a base point. Each arc has a normal orientation, indicated by a short arrow meeting with the arc transversely and is labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$ which is called the *label* of the arc.

(4) If we travel around $\partial \Delta_i$ once in the clockwise direction starting from O_i and read off the labels on arcs encountered (if we cross an arc, labelled x say, in the direction of its normal orientation, then we read x, whereas if we cross the arc in the direction of its opposite orientation, then we read x^{-1}), then we obtain a word which belongs to $\mathbf{r} \cup \mathbf{r}^{-1}$. We call this word the *label* of Δ_i. If $\mathbf{s}$ is a subset of $\mathbf{r}$, then a disc labelled by an element of $\mathbf{s} \cup \mathbf{s}^{-1}$ is called an $\mathbf{s}$-*disc*.

When we refer the discs of $\mathbb{P}$ we mean the discs $\Delta_1, \Delta_2, \cdots, \Delta_n$, not the ambient disc D^2. A closed arc which encircles no disc or arc of $\mathbb{P}$ is called a *floating circle*.

We define $\partial\mathbb{P}$ to be ∂D^2. The *label* on $\mathbb{P}$ (denoted by $W(\mathbb{P})$) is the word read off by travelling around $\partial\mathbb{P}$ once in the clockwise direction starting from O.

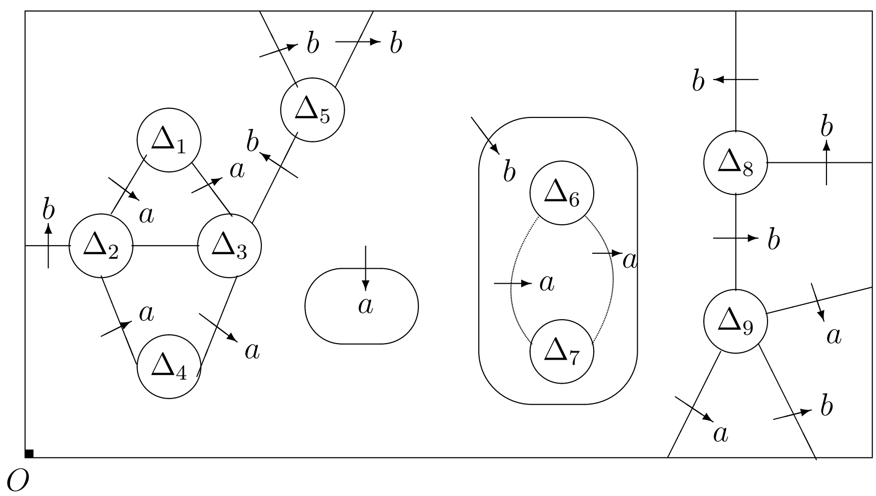
We say that $\mathbb{P}$ is *spherical* if no arcs meet $\partial\mathbb{P}$. If $\mathbb{P}$ is spherical we often omit $\partial\mathbb{P}$.

A *transverse path* γ in a picture $\mathbb{P}$ is a path in the closure of $D^2 - \bigcup_{i=1}^{n} \Delta_i$ which intersects the arcs of $\mathbb{P}$ only finitely many times. Reading off the labels on the arcs encountered while travelling along a transverse path from its initial point to its terminal point gives a word on $\mathbf{x}$ denoted $W(\gamma)$. Let γ be a simple closed transverse path in $\mathbb{P}$. The part of $\mathbb{P}$ enclosed by γ is called a *subpicture* of $\mathbb{P}$. If γ intersects no arcs, then the part of $\mathbb{P}$ enclosed by γ is called a *spherical subpicture* of $\mathbb{P}$.

A *spray* for $\mathbb{P}$ is a sequence $\underline{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ of simple transverse paths satisfying the following: for $i = 1, 2, \cdots, n$, γ_i starts at O and ends at the basepoint of Δ_i, for $1 \leqslant i < j \leqslant n$, γ_i and γ_j intersect only at O; travelling around O clockwise in $\mathbb{P}$ we encounter these transverse paths in order $\gamma_1, \gamma_2, \cdots, \gamma_n$.

Example 1.2.3 *Let $\mathcal{P} = \langle a, b\,;\, a^2, b^3, [a,b]\rangle$. Then*

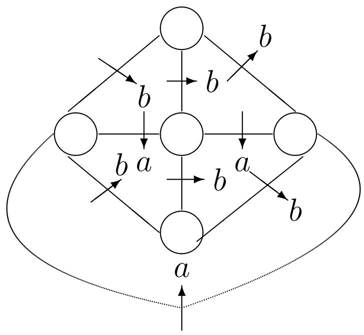
$\mathbb{P}_1$



is a picture over $\mathcal{P}$. In this picture we have nine discs $\Delta_1, \Delta_2, \cdots, \Delta_9$ with each Δ_i $(1 \leqslant i \leqslant 9)$ having a basepoint O_i on the boundary $\partial\Delta_i$. The label, for example, of the disc Δ_4 is a^2, Δ_5 is b^3 and Δ_9 is $[a,b]^{-1}$. Also, the closed arc labelled by a *is a floating circle but the closed arc labelled by b is not. We get the label on $\mathbb{P}$ is $W(\mathbb{P}) = bbbb^{-1}b^{-1}ab^{-1}a^{-1}$.*

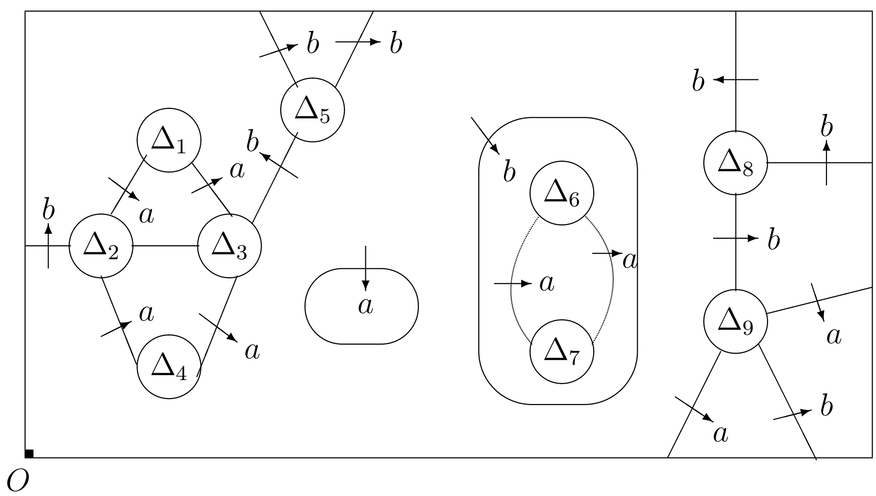
We also have an example of spherical picture $\mathbb{P}_2$ over $\mathcal{P}$ as follows.

$\mathbb{P}_2$



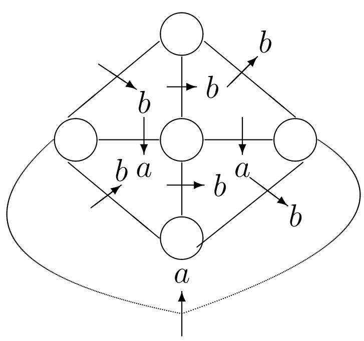
Let us fix some simple closed transverse paths γ_1, γ_2 and non-closed transverse path γ_3 into the picture $\mathbb{P}_1$ depicted as follows.

$\mathbb{P}_1$



The part enclosed by γ_1 is a spherical subpicture and the part enclosed by γ_2 is a non-spherical subpicture of $\mathbb{P}_1$. We have $W(\gamma_2) = b^2a^{-1}ba$ and $W(\gamma_3) = a^2b^{-1}$.

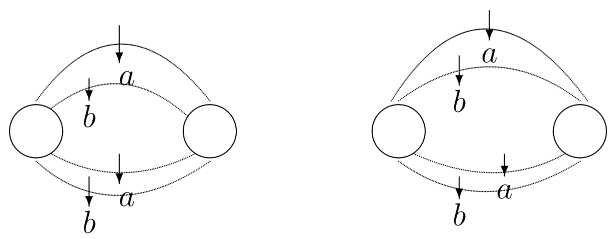
$\mathbb{P}_2$



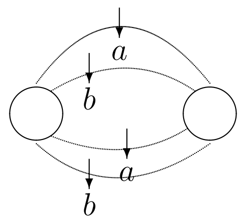
Here $\underline{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ *is a spray for* $\mathbb{P}_2$ *with* $W(\gamma_1) = 1 = W(\gamma_2)$, $W(\gamma_3) = aba^{-1}$, $W(\gamma_4) = ab^2a^{-1}$ *and* $W(\gamma_5) = a$. $\Diamond$

Throughout this thesis, each of the broken lines in a picture represents a transverse path, and they are not a part of this picture.

A *cancelling pair* in $\mathbb{P}$ is a spherical subpicture with exactly two discs whose basepoints lie in the same region. This means, for example, that



are cancelling pairs, whereas



is not.

We now introduce some elementary operations on spherical pictures as follows.

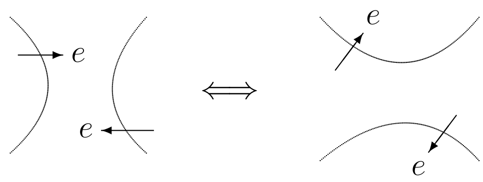
(A) Deletion of a floating circle.

$(A)^{-1}$ Insertion of a floating circle.

(B) Deletion of a cancelling pair.

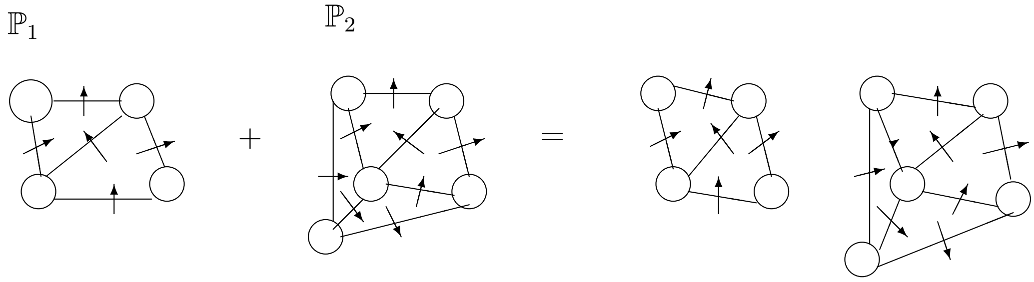
$(B)^{-1}$ Insertion of a cancelling pair.

(C) *Bridge move:*



Two spherical pictures are *equivalent* if one can be obtained from the other by a finite number of operations (A), $(A)^{-1}$, (B), $(B)^{-1}$, (C).

Let $\mathbb{P}_1$, $\mathbb{P}_2$ be spherical pictures over $\mathcal{P}$. Then the *mirror image* of $\mathbb{P}_1$ will be denoted by $-\mathbb{P}_1$, and $\mathbb{P}_1 + \mathbb{P}_2$ is the picture obtained by putting $\mathbb{P}_2$ next to $\mathbb{P}_1$. This can be illustrated as follows.

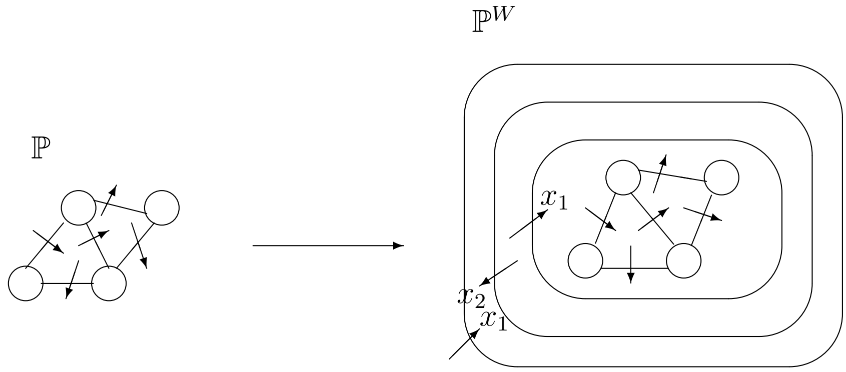


We write $\mathbb{P}_1 - \mathbb{P}_2$ for $\mathbb{P}_1 + (-\mathbb{P}_2)$. For any picture $\mathbb{P}$ over $\mathcal{P}$, $\mathbb{P} - \mathbb{P}$ is equivalent to the empty picture, and $\mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}_2 + \mathbb{P}_1$.

Let $\mathbb{P}$ be any spherical picture over $\mathcal{P}$. We denote by $< \mathbb{P} >$ the equivalence class containing $\mathbb{P}$. The set of all equivalence classes of spherical pictures over $\mathcal{P}$ forms an abelian group under the following well-defined binary operation.

$$< \mathbb{P}_1 > + < \mathbb{P}_2 > = < \mathbb{P}_1 + \mathbb{P}_2 > .$$

Let W be a word on $\mathbf{x}$, and let $\mathbb{P}$ be a spherical picture over $\mathcal{P}$. We then form a new spherical picture over $\mathcal{P}$ denoted $\mathbb{P}^W$ which is obtained from W by surrounding $\mathbb{P}$ with a collection of concentric arcs with total label W. Then this can be illustrated as follows (with $W = x_1 x_2^{-1} x_1$).



There is a well-defined $G(\mathcal{P})$-action on equivalence classes of spherical pictures given by

$$\overline{W}\,.<\mathbb{P}>=<\mathbb{P}^W> \quad (\overline{W}\in G).$$

We then obtain a $\mathbb{Z}G(\mathcal{P})$-module $\pi_2(\mathcal{P})$ called the *second homotopy module* of $\mathcal{P}$.

There is an embedding μ of $\pi_2(\mathcal{P})$ into the free module $\bigoplus_{R\in\mathbf{r}}\mathbb{Z}G(\mathcal{P})e_R$ defined as follows (see also [11], [13], [49]).

Let $<\mathbb{P}>\in\pi_2(\mathcal{P})$ and suppose that $\mathbb{P}$ has discs $\Delta_1,\,\Delta_2,\,\cdots,\,\Delta_n$ with the label $R_1^{\varepsilon_1},\,R_2^{\varepsilon_2},\,\cdots,\,R_n^{\varepsilon_n}$ respectively ($R_i\in\mathbf{r}$, $\varepsilon_i=\pm1$, $i=1,2,\cdots,n$). Let $\underline{\gamma}=(\gamma_1,\,\cdots,\,\gamma_n)$ be a spray, as defined previously. Recall that $W(\gamma_i)$ is the label on γ_i which represents an element of G. Then

$$\mu(<\mathbb{P}>)=\sum_{i=1}^{n}\varepsilon_i\overline{W(\gamma_i)}e_{R_i}.$$

We often write $\mu(\mathbb{P})$ instead of $\mu(<\mathbb{P}>)$.

Example 1.2.3 (continued) *For the spherical picture $\mathbb{P}_2$, we get*

$$\mu(\mathbb{P}_2)=(-1+\overline{a})e_{b^3}-(1+\overline{aba^{-1}}+\overline{ab^2a^{-1}})e_{[a,b]}.$$

$\Diamond$

For each spherical picture $\mathbb{P}$ over $\mathcal{P}$ and for each $R\in\mathbf{r}$, let $\lambda_{\mathbb{P},R}$ be the coefficients of e_R in $\mu(\mathbb{P})$. Let $I_2(\mathcal{P})$ be the 2-sided ideal in $\mathbb{Z}G$ generated by the set

$$\{\lambda_{\mathbb{P},R}:\mathbb{P}\text{ is a spherical picture, }R\in\mathbf{r}\}.$$

This ideal is called the *second Fox ideal* of $\mathcal{P}$. The concept of Fox ideals has been discussed in [43], [44]. In fact we need this concept for Theorem 1.2.17 below which is a test of minimality of group presentations.

Let us consider a collection $\mathbf{X}$ of spherical pictures over $\mathcal{P}$. We introduce two further operations on spherical pictures.

(D) Delete a spherical subpicture which is a copy of some elements of $\mathbf{X} \cup -\mathbf{X}$.

$(D)^{-1}$ The opposite of (D).

Two spherical pictures will be said to be *equivalent (relative to* $\mathbf{X}$*)* if one can be transformed to the other by a finite number of operations $(A)^{\pm 1}$, $(B)^{\pm 1}$, (C) and $(D)^{\pm 1}$. Then, by [49] (see Theorem 2.6*, Corollary 1), we have

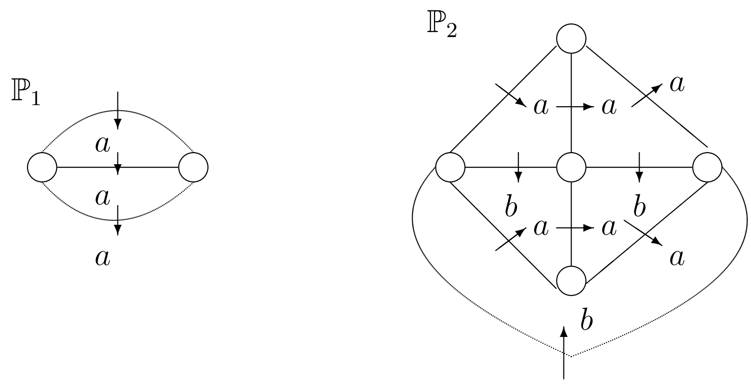
Theorem 1.2.4 *The elements* $< \mathbb{P} >$ $(\mathbb{P} \in \mathbf{X})$ *generate* $\pi_2(\mathcal{P})$ *if and only if every spherical picture is equivalent to the empty picture (relative to* $\mathbf{X}$*).*

We say that $\mathbf{X}$ *generates* $\pi_2(\mathcal{P})$ (or $\mathbf{X}$ is a set of *generating pictures*) if the elements $< \mathbb{P} >$ $(\mathbb{P} \in \mathbf{X})$ generate $\pi_2(\mathcal{P})$.

It can be shown that if $\mathbf{X}$ is a set of generating pictures, then $I_2(\mathcal{P})$ is generated (as a 2-sided ideal) by

$$\{\lambda_{\mathbb{P},R} : \mathbb{P} \in \mathbf{X}, R \in \mathbf{r}\}.$$

Example 1.2.5 *Let* $G = \mathbb{Z}_3 \oplus \mathbb{Z}$ *be defined by* $\mathcal{P} = \langle a, b \,;\, a^3, [a, b] \rangle$*. Then, by* [5]*,* $\pi_2(\mathcal{P})$ *is generated by*



Then, $\mu(\mathbb{P}_1) = (1-\overline{a})e_{a^3}$ *and* $\mu(\mathbb{P}_2) = (\overline{b}-1)e_{a^3} + (1+\overline{bab^{-1}}+\overline{ba^2b^{-1}})e_{[a,b]}$. *Thus,* $I_2(\mathcal{P})$ *is generated by* $\{\overline{b}-1,\ 1+\overline{bab^{-1}}+\overline{ba^2b^{-1}},\ 1-\overline{a}\}$. *Note that* $\overline{bab^{-1}} = \overline{a}$ *and* $\overline{ba^2b^{-1}} = \overline{a}^2$.

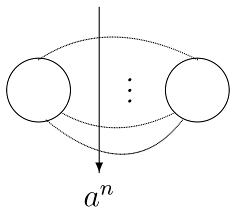
1.2.3 Aspherical and Cockcroft presentations

Definition 1.2.6 *Let* $\mathcal{P}$ *be as in* (1.2). *Then* $\mathcal{P}$ *is said to be* **aspherical** *if* $\pi_2(\mathcal{P}) = 0$. *A group* G *is said to be aspherical if it is defined by an aspherical presentation.*

We remark that all free groups and torsion free one-relator groups [45] are aspherical. Some other examples of aspherical presentations can be found, for instance in [12], [16], [49].

Definition 1.2.7 *Let* $\mathcal{P}$ *be as in* (1.2). *Then* $\mathcal{P}$ *is said to be* **combinatorial aspherical (CA)** *if* $\pi_2(\mathcal{P})$ *is generated by a set of pictures containing exactly two discs. A group* G *is said to be combinatorial aspherical* (CA) *if it can be defined by a* CA *presentation.*

Example 1.2.8 *Let* $\mathcal{P} = \langle a\ ;\ a^n \rangle$ *be a presentation of cyclic group of order* n. *It is known that* $\pi_2(\mathcal{P})$ *is generated by the following single picture.*



Therefore $\mathcal{P}$ *is* CA. $\Diamond$

One-relator groups with torsion are CA (but not aspherical) (see [45]). Some other examples of combinatorial aspherical presentations can also be found, for example, in [12], [16], [30], [31], [49].

For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $exp_R(\mathbb{P})$ is the number of discs of $\mathbb{P}$ labelled by R, minus the number of discs labelled by R^{-1}. We remark that if pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent, then $exp_R(\mathbb{P}_1) = exp_R(\mathbb{P}_2)$ for all $R \in \mathbf{r}$.

Definition 1.2.9 *Let $\mathcal{P}$ be as in* (1.2)*, and let n be a non-negative integer. Then $\mathcal{P}$ is said to be n-**Cockcroft** if $exp_R(\mathbb{P}) \equiv 0 \ (mod\ n)$, (where congruence $(mod\ 0)$ is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. A group G is said to be n-Cockcroft if it admits an n-Cockcroft presentation.*

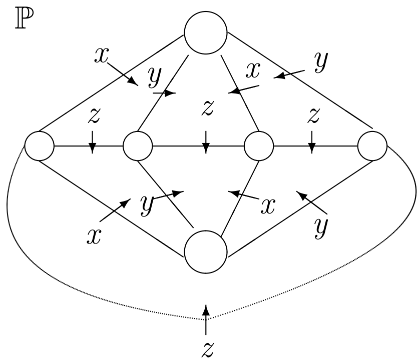
Remark 1.2.10 *To verify that the n-Cockcroft property holds, it is enough to check for pictures $\mathbb{P} \in \mathbf{X}$, where $\mathbf{X}$ is a set of generating pictures.*

The 0-Cockcroft property is usually just called Cockcroft.

In practice, we usually take n to be 0 or a prime p.

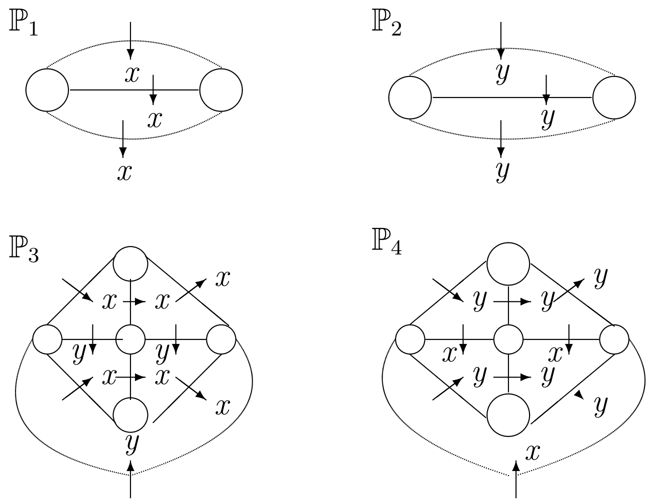
The Cockcroft property has received considerable attention in [22], [25], [26], [27] and [41]. The p-Cockcroft property has been discussed for example in [41].

Example 1.2.11 *Let $\mathcal{P} = \langle x,\ y,\ z\ ;\ [x,y],\ [x,z],\ [y,z]\rangle$. Then one may refer to* [5] *to show that $\pi_2(\mathcal{P})$ is generated by*



Now since $exp_{[x,y]}(\mathbb{P}) = exp_{[x,z]}(\mathbb{P}) = exp_{[y,z]}(\mathbb{P}) = 1 - 1 = 0$ then $\mathcal{P}$ is Cockcroft.

Example 1.2.12 *Let $\mathcal{P} = \langle x,\ y\ ;\ ;x^3,\ y^3,\ [x,y]\rangle$. Then, by* [5]*, $\pi_2(\mathcal{P})$ is generated by*



Then $exp_{x^3}(\mathbb{P}_1) = exp_{y^3}(\mathbb{P}_2) = exp_{x^3}(\mathbb{P}_3) = exp_{y^3}(\mathbb{P}_4) = 1 - 1 = 0$*,* $exp_{[x,y]}(\mathbb{P}_3) = 3$ *and* $exp_{[x,y]}(\mathbb{P}_4) = -3$*. Thus* $\mathcal{P}$ *is* 3*-Cockcroft.*

Note that

$$\text{Aspherical} \Rightarrow \text{CA} \Rightarrow \text{Cockcroft} \Rightarrow n\text{-Cockcroft} \quad (n \in \mathbb{Z}^+).$$

1.2.4 Efficiency of group presentations

Let $\mathcal{P}$ be as in (1.2). Then we define the *Euler characteristic* of $\mathcal{P}$ as follows.

$$\chi(\mathcal{P}) = 1 - |\mathbf{x}| + |\mathbf{r}|.$$

Let

$$\delta(G) = 1 - rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G)), \tag{1.3}$$

where $rk(\ )$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(\ )$ means the minimal number of generators. Then it is known (see [5], [10], [23]) that for the presentation $\mathcal{P}$, it is always true that

$$\chi(\mathcal{P}) \geqslant \delta(G).$$

Then we define

$$\chi(G) = min\{\chi(\mathcal{P}) : \mathcal{P} \text{ a finite presentation for } G\}.$$

We should remark that some authors consider just

$$-|\mathbf{x}| + |\mathbf{r}|,$$

and call this the *deficiency* of the presentation $\mathcal{P}$, denote by $def(\mathcal{P})$. The deficiency of a group G, denote by $def(G)$, is then taken to be the minimum deficiencies of all finite presentations of G. Clearly

$$1 + def(\mathcal{P}) = \chi(\mathcal{P}),$$
$$1 + def(G) = \chi(G).$$

Definition 1.2.13 *Let G be a group.*

i) A presentation $\mathcal{P}_0$ for G is called **minimal** *if*

$$\chi(\mathcal{P}_0) \leqslant \chi(\mathcal{P}),$$

for all presentations $\mathcal{P}$ of G.

ii) A presentation $\mathcal{P}$ is called **efficient** *if*

$$\chi(\mathcal{P}) = \delta(G).$$

iii) G is called **efficient** *if*

$$\chi(G) = \delta(G).$$

Lemma 1.2.14 *(i) If $\chi(G) \leqslant 0$ then G must be infinite.*

(ii) If G is finite cyclic then $\chi(G) = 1$.

Proof.

(i) It can be found, for example in [46] or [47], that for a presentation of the group G, if the number of generators is greater than the number of relators then G is infinite.

(ii) Let G be a cyclic group of order n with the presentation $\mathcal{P} = \langle x \ ; \ x^n \rangle$. By definition, $\chi(G) \leqslant \chi(\mathcal{P})$, that is $\chi(G) \leqslant 1$. But, by *(i)*, $\chi(G)$ cannot be less than 1,

otherwise G would be infinite cyclic, a contradiction. Hence $\chi(G)$ must be equal to 1, as required. □

Examples of efficient groups are finitely generated abelian groups (Epstein [23]), fundamental groups of closed 3-manifolds [23]; also finite groups with balanced presentations (such finite groups have trivial Schur multiplier [28]). Finite metacyclic groups are efficient. This was shown by Beyl [8] and Wamsley [59]. Infinite metacyclic groups however need not be efficient, a result due to Baik and Pride [5] (see also [3]). In [28] Harlander proved that a finitely presented group embeds into an efficient group. For more references on the subject of efficiency see Baik, Pride [4], Beyl, Rosenberger [9], Champbell, Robertson, Williams [14] (and [15]), Harlander [29], Johnson, Robertson [37], Kenne [39], Robertson, Thomas, Wotherspoon [53].

The following result which is essentially due to Epstein [23] can be found in [41, Theorem 2.1].

Theorem 1.2.15 *Let $\mathcal{P}$ be as in* (1.2). *Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

As a consequence of the above theorem, we have

Corollary 1.2.16 *Let $\mathcal{P}$ be as in* (1.2). *If $\mathcal{P}$ is Cockcroft then $\mathcal{P}$ is efficient.*

Not all finitely presented groups are efficient.

B.H.Neumann [48] asked whether a finite group G with $\delta(G) = 1$ must be efficient. Swan [57] gave examples (of finite metabelian groups) showing this is not the case. These were the first examples of inefficient groups. In [61], Wiegold produced a different construction to the same end, and then Neumann added a slight modification to reduce the number of generators. In [42], Kovacs generalized both the above constructions, and he showed how to construct more inefficient finite groups (including some perfect groups) whose Schur multiplicator is trivial. In [53], Robertson, Thomas and Wotherspoon examined a class of groups, orginally introduced by Coxeter. By using a symmetric presentation, they showed that groups in this class are inefficient. They also proved that every finite simple group can be embedded into a finite inefficient group.

Lustig [44] gave the first example of a torsion-free inefficient group. Other examples were found by Baik (see [3]), using generalized graph products. In [4], Baik and Pride gave sufficient conditions for a Coxeter group to be efficient. They also found a family of inefficient Coxeter group $G_{n,k}$ ($n \geqslant 4$, k an odd integer). For a fixed k,

$$\chi(G_{n,k}) - \delta(G_{n,k}) \xrightarrow{n} \infty.$$

We remark that there is no algorithm to decide for any finitely presented group whether or not the group is efficient (see [1]).

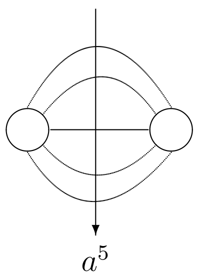
The next result, due to Lustig [44] (see also [41]) gives a method of showing that a presentation is minimal.

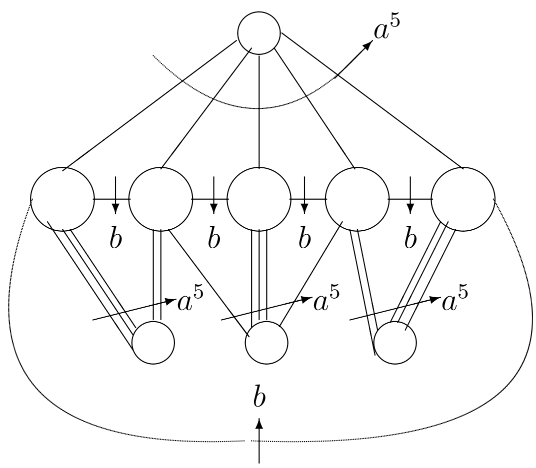
Theorem 1.2.17 ([44]) *Let G be a group with the presentation $\mathcal{P}$ as in* (1.2). *If there is a ring homomorphism ϕ from $\mathbb{Z}G$ into the matrix ring of all $k \times k$-matrices ($k \geqslant 1$) over some commutative ring A with* 1*, such that $\phi(1) = 1$, and if ϕ maps the second Fox ideal $I_2(\mathcal{P})$ to* 0*, then $\mathcal{P}$ is minimal.*

Example 1.2.18 ([3]) *Let G be a group defined by the presentation*

$$\mathcal{P} = \left\langle a,\, b\, ;\; a^5,\, aba^{-3}b^{-1} \right\rangle .$$

$\pi_2(\mathcal{P})$ is generated by





It is clear that $\mathcal{P}$ is not p-Cockcroft for any prime p, and hence not efficient by Theorem 1.2.15*. We will show that $\mathcal{P}$ is minimal and so there could not be an efficient presen-*

tation which defines the group G. Thus we can conclude that G is not p-Cockcroft for any prime p.

From the above pictures, $I_2(\mathcal{P})$ is generated by

$$1 - \overline{a},\ 1 + \overline{a} + \overline{a^2} + \overline{a^3} + \overline{a^4},\ 3\overline{b} - 1.$$

Let $< x >$ be an infinite cyclic group and consider the ring homomorphism

$$\mathbb{Z}G \longrightarrow \mathbb{Z} < x >$$

arising from the group homomorphism defined by

$$a \longmapsto 1,\ b \longmapsto x.$$

If we consider

$$\mathbb{Z} < x > \longrightarrow \mathbb{Z}_5$$

by sending all integer coefficients to their congruence modulo 5 and sending x to the congruence class of 2, then the mapping

$$\mathbb{Z}G \longrightarrow \mathbb{Z} < x > \longrightarrow \mathbb{Z}_5$$

sends the generators of $I_2(\mathcal{P})$ to 0 and 1 to 1. Hence, by Theorem 1.2.17, $\mathcal{P}$ is minimal.

$\Diamond$

1.3 Monoid presentations

A *monoid presentation*

$$\mathcal{P} = [\mathbf{y}\ ;\ \mathbf{s}] \tag{1.4}$$

is a pair, where $\mathbf{y}$ is a set (the *generating symbols*) and each $S \in \mathbf{s}$ (a *relation*) is an ordered pair (S_+, S_-), where S_+ and S_- are distinct, positive words on $\mathbf{x}$. We remark that one of S_+, S_- may be the empty positive word. We usually write $S : S_+ = S_-$. Once again, we say that $\mathcal{P}$ is finite if $\mathbf{y}$ and $\mathbf{s}$ are both finite.

Throughout this thesis, all monoid presentations will be assumed to be finite unless stated otherwise.

In order to define a monoid associated with $\mathcal{P}$ we introduce the following elementary operation on positive words on $\mathbf{y}$. Let W be a positive word on $\mathbf{y}$.

$(\bullet)$: If W contains a subword S_ε, where $\varepsilon = \pm 1$, $S_+ = S_- \in \mathbf{s}$, then replace it by S_-.

Two positive words W_1, W_2 on $\mathbf{y}$ are *equivalent (relative to $\mathcal{P}$)*, denoted $W_1 \sim_{\mathcal{P}} W_2$, if there is a finite chain of elementary operations of type $(\bullet)$ leading from W_1 to W_2. This is an equivalence relation on the set of all positive words on $\mathbf{y}$. Let $[W]_{\mathcal{P}}$ denote the equivalence class containing W. A multiplication can be defined on equivalence classes by $[W_1]_{\mathcal{P}}.[W_2]_{\mathcal{P}} = [W_1W_2]_{\mathcal{P}}$. It is easy to check that this multiplication is well-defined. The set of all equivalence classes together with this multiplication form a monoid, the *monoid defined by* $\mathcal{P}$, denoted $M(\mathcal{P})$. The identity in $M(\mathcal{P})$ is $[1]_{\mathcal{P}}$.

For a positive word W on $\mathbf{y}$, we will denote the element $[W]_{\mathcal{P}}$ by $\overline{W}$.

1.3.1 Fox derivations

Let $\hat{F}(\mathbf{y})$ be the free monoid on $\mathbf{y}$. For a fixed $y \in \mathbf{y}$, we define a function

$$\frac{\partial}{\partial y} : \hat{F}(\mathbf{y}) \longrightarrow \mathbb{Z}\hat{F}(\mathbf{y})$$

as follows. Let $W \in \hat{F}(\mathbf{y})$ and write

$$W = W_0yW_1y \cdots W_{r-1}yW_r, \tag{1.5}$$

where $r \geqslant 1$, $W_0, W_1, \cdots, W_r$ are positive words on $\mathbf{y} - \{y\}$. Then

$$\frac{\partial W}{\partial y} = \sum_{i=1}^{r} W_0yW_1y \cdots W_{i-1}.$$

We then extend $\dfrac{\partial}{\partial y}$ to a function

$$\frac{\partial}{\partial y} : \mathbb{Z}\hat{F}(\mathbf{y}) \longrightarrow \mathbb{Z}\hat{F}(\mathbf{y})$$

given by

$$\frac{\partial}{\partial y}(n_1W_1 + n_2W_2 + \cdots + n_rW_r) = \sum_{i=1}^{r} n_i \frac{\partial W_i}{\partial y},$$

where $r \geqslant 0$, $n_1, \cdots, n_r \in \mathbb{Z}$, $W_1, \cdots, W_r \in \hat{F}(\mathbf{y})$.

Let M be a monoid with the presentation $\mathcal{P}$, as in (1.4). We have the natural ring homomorphism

$$\mathbb{Z}\hat{F}(\mathbf{y}) \longrightarrow \mathbb{Z}M$$

induced by the monoid homomorphism

$$\hat{F}(\mathbf{y}) \longrightarrow M, \quad W \longmapsto \overline{W}.$$

We write $\dfrac{\partial^M}{\partial y}$ or $\dfrac{\partial^{\mathcal{P}}}{\partial y}$ for the composition

$$\mathbb{Z}\hat{F}(\mathbf{y}) \xrightarrow{\frac{\partial}{\partial y}} \mathbb{Z}\hat{F}(\mathbf{y}) \longrightarrow \mathbb{Z}M.$$

Thus, for $W \in \hat{F}(\mathbf{y})$ as in (1.5),

$$\frac{\partial^M W}{\partial y} = \sum_{i=1}^{r} \overline{W_0 y W_1 y \cdots W_{i-1}}.$$

Let

$$aug : \mathbb{Z}M \longrightarrow \mathbb{Z}, \quad m \longmapsto 1$$

be the augmentation map. Then we have

Lemma 1.3.1 *For a fixed $y \in \mathbf{y}$,*

$$aug(\frac{\partial^M W}{\partial y}) = L_y(W).$$

Proof.

$$\begin{aligned} aug(\frac{\partial^M W}{\partial y}) &= aug(\sum_{i=1}^{r} \overline{W_0 y W_1 y \cdots W_{i-1}}) \\ &= r \\ &= L_y(W) \text{ since the number of all occurences} \\ &\quad \text{of } y \text{ in } W \text{ is the length of } y \text{ in } W. \end{aligned}$$

□

1.3.2 Pictures over monoid presentations

The material used in this section may also be found in [50], [51].

Let $\mathcal{P}$ be a monoid presentation, as in (1.4), and let $\hat{F}(\mathbf{y})$ be the free monoid on $\mathbf{y}$. If we have an element

$$W = US_\varepsilon V \quad (U, V \in \hat{F}(\mathbf{y}),\, S \in \mathbf{s},\, \varepsilon = \pm 1)$$

of $\hat{F}(\mathbf{y})$, then we can replace S_ε by $S_{-\varepsilon}$ to get a word

$$W' = US_{-\varepsilon}V.$$

This can be represented by a geometric object called an *atomic (monoid) picture*

$$\mathbb{A} = (U, S, \varepsilon, V)$$

as depicted in Figure 1.1.

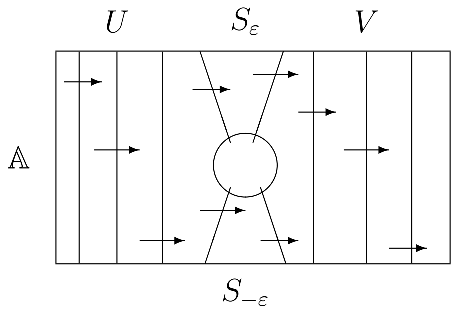


Figure 1.1:

We remark that the disc labelled by S in an atomic picture $\mathbb{A}$ is said to be *positive* if $\varepsilon = 1$, and said to be *negative* if $\varepsilon = -1$.

We have a graph Γ $(= \Gamma(\mathcal{P}))$ associated with $\mathcal{P}$, called the Squier graph, which is defined as follows. The *vertex set* is $\hat{F}(\mathbf{y})$, and the *edge set* is the collection of all atomic monoid pictures. For an orientation of Γ we will take all edges $(U, S, +1, V)$. For an atomic picture $\mathbb{A}$, as in Figure 1.1, the word we read off by travelling along the top of the atomic picture from left to right gives the *initial* function, denoted by

$$\iota(\mathbb{A}) = US_\varepsilon V,$$

and the word we read off by travelling along the bottom gives the *terminal* function, denoted by

$$\tau(\mathbb{A}) = US_{-\varepsilon}V.$$

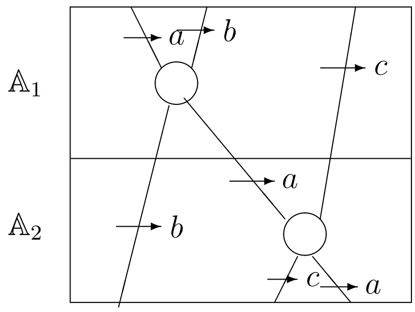
Also, the *mirror image* of $\mathbb{A}$ is denoted by

$$\mathbb{A}^{-1} = (U, S, -\varepsilon, V).$$

A *path*

$$\mathbb{P} = \mathbb{A}_1\mathbb{A}_2 \cdots \mathbb{A}_n \qquad (1.6)$$

(where each $\mathbb{A}_i$ is an atomic picture for $i = 1, 2, \cdots, n$) in Γ will also be called a *monoid picture* over $\mathcal{P}$. If $\iota(\mathbb{A}_1) = \tau(\mathbb{A}_n)$ then $\mathbb{P}$ is called a *spherical monoid picture* over $\mathcal{P}$, otherwise $\mathbb{P}$ is called a *non-spherical monoid picture* over $\mathcal{P}$. For example,



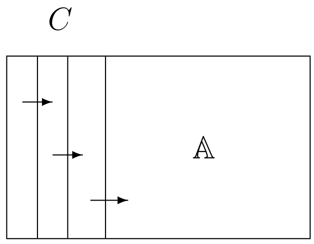
is a non-spherical monoid picture, since $\iota(\mathbb{A}_1) \neq \tau(\mathbb{A}_2)$. (For an example of spherical monoid picture see Figure 1.2).

Note that we also have the term *subpicture* (that is, subpath) of monoid pictures. For example, the non-spherical picture depicted in the above figure is a subpicture of the spherical monoid picture as shown in Figure 1.2.

There is a left action of $\hat{F}(\mathbf{y})$ on Γ defined as follows. Let $C \in \hat{F}(\mathbf{y})$.

i) Let W be a vertex of Γ. Then we define $C.W$ to be CW (product in $\hat{F}(\mathbf{y})$).

ii) Let $\mathbb{A}$, as in Figure 1.1, be an edge of Γ. Then $C.\mathbb{A} = (CU, S, \varepsilon, V)$ and this can be illustrated by



We can define a similar right action of $\hat{F}(\mathbf{y})$ on Γ. The left and right actions of $\hat{F}(\mathbf{y})$ on Γ extends to actions on pictures. That is, if $\mathbb{P}$ is a picture as in (1.6), and if $W, V \in \hat{F}(\mathbf{y})$ then

$$W.\mathbb{P}.V = (W.\mathbb{A}_1.V)(W.\mathbb{A}_2.V)\cdots(W.\mathbb{A}_n.V).$$

Example 1.3.2 *Let* $\mathcal{P} = [a,\ b,\ c\ ;\ ab = ba,\ bc = cb,\ ca = ac]$*, and let*

$$\mathbb{A}_1 = (1, ab = ba, +1, c), \qquad \mathbb{A}_2 = (b, ac = ca, -1, 1),$$
$$\mathbb{A}_3 = (1, bc = cb, +1, a), \qquad \mathbb{A}_4 = (c, ba = ab, -1, 1),$$
$$\mathbb{A}_5 = (1, ca = ac, +1, b), \qquad \mathbb{A}_6 = (a, cb = bc, -1, 1).$$

Then $\tau(\mathbb{A}_i) = \iota(\mathbb{A}_{i+1})$ *for* $i = 1, 2, \cdots, 6$*, and* $\iota(\mathbb{A}_1) = \tau(\mathbb{A}_6) = abc$*. So* $\mathbb{P} = \mathbb{A}_1\mathbb{A}_2\cdots\mathbb{A}_6$ *is a spherical monoid picture (see Figure* 1.2.*(i)). Now by a left action by* a *and a right action by* c*, we obtain another spherical monoid picture. This can be illustrated as in Figure* 1.2.*(ii).* $\diamondsuit$

We now introduce some operations on spherical monoid pictures. Let $\mathbb{A}$, $\mathbb{B}$ be atomic pictures.

(A) Delete an inverse pair $\mathbb{A}\mathbb{A}^{-1}$.

$(A)^{-1}$ The opposite of (A).

(B) Replace a subpicture $(\mathbb{A}\,.\,\iota(\mathbb{B}))(\tau(\mathbb{A})\,.\,\mathbb{B})$ by $(\iota(\mathbb{A})\,.\,\mathbb{B})(\mathbb{A}\,.\,\tau(\mathbb{B}))$ or vice versa (see Figure 1.3).

Two spherical monoid pictures are said to be *equivalent* if one can be obtained from the other by a finite number of operations $(A)^{\pm 1}, (B)$.

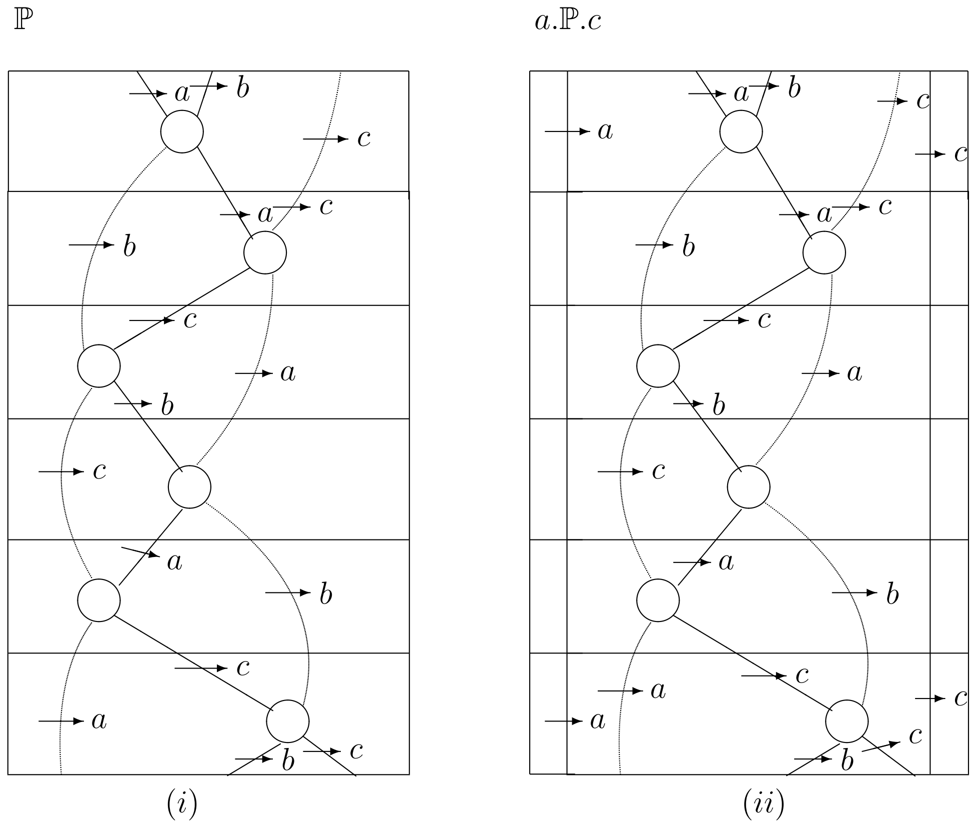


Figure 1.2:

The graph Γ with the above equivalence relation on paths, is called the *Squier complex* of $\mathcal{P}$ denoted by $\mathcal{D}(\mathcal{P})$ (see, for example, [51]). (More formally, $\mathcal{D}(\mathcal{P})$ consist of the graph Γ, together with defining paths which are all the closed paths

$$[\mathbb{A}, \mathbb{B}] = (\mathbb{A} \,.\, \iota(\mathbb{B}))(\tau(\mathbb{A}) \,.\, \mathbb{B})(\mathbb{A}^{-1} \,.\, \tau(\mathbb{B}))(\iota(\mathbb{A}) \,.\, \mathbb{B}^{-1}),$$

($\mathbb{A}$, $\mathbb{B}$ are atomic pictures), as shown in Figure 1.3.)

Let $\mathbf{Y}$ be a set of spherical monoid pictures. We introduce two further operations on spherical monoid pictures as follows.

(C) Delete subpictures of the form $W \,.\, \mathbb{P}^{\pm 1} \,.\, V$ ($\mathbb{P} \in \mathbf{Y}$, $W, V \in \hat{F}(\mathbf{y})$).

$(C)^{-1}$ The opposite of (C).

Two spherical monoid pictures will be said to be *equivalent* (*relative to* $\mathbf{Y}$) if one can be transformed to other by a finite number of operations $(A)^{\pm 1}$, (B), $(C)^{\pm 1}$.

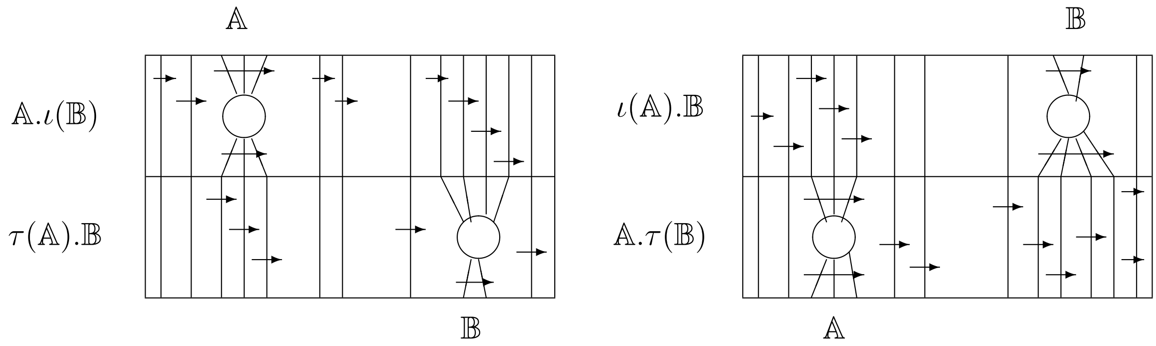


Figure 1.3:

By [51, Theorem 5.1], we say that $\mathbf{Y}$ is a *trivialiser* of $\mathcal{D}(\mathcal{P})$ if every spherical picture is equivalent to an empty picture (relative to $\mathbf{Y}$). Some examples and the details of the trivialiser can be found in [20], [51], [52], [56], [60]. In Section 4.3.4, as an example of this, we will give a trivialiser set of the Squier complex of a presentation of the semi-direct product of any two monoids, as found by Wang (see [60]).

Let M be a monoid with the presentation $\mathcal{P}$, as in (1.4). Let

$$P^{(l)} = \bigoplus_{S\in\mathbf{s}} \mathbb{Z}Me_S \ \text{ and } \ P^{(r)} = \bigoplus_{S\in\mathbf{s}} f_S\mathbb{Z}M$$

be the free left and right $\mathbb{Z}M$-modules with bases

$$\{e_S : S \in \mathbf{s}\} \ \text{ and } \ \{f_S : S \in \mathbf{s}\},$$

respectively. For an atomic picture $\mathbb{A} = (U, S, \varepsilon, V)$ ($U, V \in \hat{F}(\mathbf{y})$, $S \in \mathbf{s}$, $\varepsilon = \pm 1$), as in Figure 1.1, we define

$$eval^{(l)}(\mathbb{A}) = \varepsilon\overline{U}e_S \in P^{(l)} \ \text{ and } \ eval^{(r)}(\mathbb{A}) = \varepsilon f_S\overline{V} \in P^{(r)},$$

where $\overline{U}$, $\overline{V} \in M(\mathcal{P}) \cong M$. For any spherical monoid picture $\mathbb{P}$, as in (1.6), we define

$$\begin{aligned} eval^{(l)}(\mathbb{P}) &= \sum_{i=1}^{n} eval^{(l)}(\mathbb{A}_i) \in P^{(l)}, \\ eval^{(r)}(\mathbb{P}) &= \sum_{i=1}^{n} eval^{(r)}(\mathbb{A}_i) \in P^{(r)}. \end{aligned}$$

We let $\lambda_{\mathbb{P},S}$ be the coefficient of e_S in $eval^{(l)}(\mathbb{P})$, so we can write

$$eval^{(l)}(\mathbb{P}) = \sum_{S\in\mathbf{s}} \lambda_{\mathbb{P},S}e_S \in P^{(l)}.$$

Similarly, we let $\eta_{\mathbb{P},S}$ be the coefficient in $eval^{(r)}(\mathbb{P})$, so

$$eval^{(r)}(\mathbb{P}) = \sum_{S \in \mathbf{s}} f_S \eta_{\mathbb{P},S} \in P^{(r)}.$$

Example 1.3.2 (continued) *Let*

$$R : ab = ba, \;\; S : bc = cb, \;\; T : ca = ac.$$

Then we have

$$\begin{aligned} &eval^{(l)}(\mathbb{A}_1) = e_R, && eval^{(l)}(\mathbb{A}_2) = -\bar{b}e_T, && eval^{(l)}(\mathbb{A}_3) = e_S, \\ &eval^{(l)}(\mathbb{A}_4) = -\bar{c}e_R, && eval^{(l)}(\mathbb{A}_5) = e_T, && eval^{(l)}(\mathbb{A}_6) = -\bar{a}e_S. \end{aligned}$$

Thus

$$eval^{(l)}(\mathbb{P}) = \lambda_{\mathbb{P},R}e_R + \lambda_{\mathbb{P},S}e_S + \lambda_{\mathbb{P},T}e_T,$$

where

$$\lambda_{\mathbb{P},R} = 1 - \bar{c}, \;\; \lambda_{\mathbb{P},S} = 1 - \bar{a}, \;\; \lambda_{\mathbb{P},T} = 1 - \bar{b}.$$

$\diamondsuit$

Definition 1.3.3 *Let $I_2^{(l)}(\mathcal{P})$, $I_2^{(r)}(\mathcal{P})$ be the 2-sided ideals of $\mathbb{Z}M$ generated by the sets*

$$\{\lambda_{\mathbb{P},S} : \mathbb{P} \text{ is a spherical monoid picture}, S \in \mathbf{s}\},$$

$$\{\eta_{\mathbb{P},S} : \mathbb{P} \text{ is a spherical monoid picture}, S \in \mathbf{s}\},$$

respectively. Then these ideals are called the **second Fox ideals** *of $\mathcal{P}$.*

Remark 1.3.4 *If $\mathbf{Y}$ is a trivializer of $\mathcal{D}(\mathcal{P})$ then $I_2^{(l)}(\mathcal{P})$ and $I_2^{(r)}(\mathcal{P})$ are generated (as 2-sided ideals) by the sets*

$$\{\lambda_{\mathbb{P},S} : \mathbb{P} \in \mathbf{Y}, S \in \mathbf{s}\} \;\; \text{and} \;\; \{\eta_{\mathbb{P},S} : \mathbb{P} \in \mathbf{Y}, S \in \mathbf{s}\},$$

respectively.

Example 1.3.5 *Let $\mathcal{P} = [a,\, b,\, c\,;\; aba = ba^2,\, ac = ca^3,\, bc = cb]$. Then, by [60], a trivialiser $\mathbf{Y}$ of $\mathcal{D}(\mathcal{P})$ can be taken to contains a single monoid picture $\mathbb{P}$ depicted in Figure 1.4. Let*

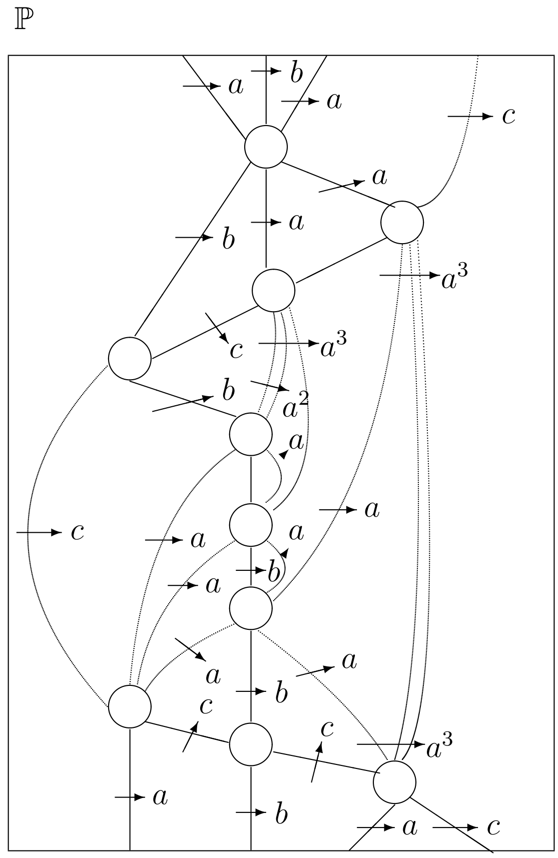


Figure 1.4:

$$R : aba = ba^2,\ \ S : ac = ca^3,\ \ T : bc = cb.$$

Then

$$eval^{(l)}(\mathbb{P}) = \lambda_{\mathbb{P},R}e_R + \lambda_{\mathbb{P},S}e_S + \lambda_{\mathbb{P},T}e_T,$$

where

$$\lambda_{\mathbb{P},R} = 1 - \overline{c(1 + a + a^2)},\ \ \lambda_{\mathbb{P},S} = \overline{b} - 1,\ \ \lambda_{\mathbb{P},T} = 1 - \overline{a}.$$

Thus, by the above remark, the second Fox ideal $I_2^{(l)}(\mathcal{P})$ is generated by the set $\{1 - \overline{c(1 + a + a^2)}, \overline{b} - 1, 1 - \overline{a}\}$. $\Diamond$

Note that we need the second Fox ideal concept for Theorem 1.3.14 (see below).

1.3.3 Aspherical and Cockcroft monoid presentations

Definition 1.3.6 *Let $\mathcal{P}$ be as in* (1.4)*. Then $\mathcal{P}$ is said to be* **aspherical** *if there are no non-trivial spherical monoid pictures over $\mathcal{P}$.*

Note that all free monoids are aspherical. In [34, Section 5] Ivanov proved that if M is a one-relator monoid, with relator S, say and if $\iota(S_+) \neq \tau(S_-)$ (or $\tau(S_+) \neq \iota(S_-)$) then M is aspherical. Some other examples of asphericity can be found, for instance in [51, Section 7], [40], [34] and [17].

Definition 1.3.7 *Let $\mathcal{P}$ be as in* (1.4)*. Then $\mathcal{P}$ is said to be* **combinatorial aspherical (CA)** *if $\mathcal{P}$ has a trivialiser set $\mathbf{Y}$ such that every element of $\mathbf{Y}$ contains exactly two discs. Also, a monoid M is said to be combinatorial aspherical if it can be defined by a* (CA) *presentation.*

In Chapter 4 we will use that all finite cyclic monoids are (CA) (see Lemma 4.2.13). See [51, Section 7] for further discussion on the combinatorial asphericity.

For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $S \in \mathbf{s}$, the *exponent sum* of S in $\mathbb{P}$ is the number of positive discs labelled by S, minus the number of negative discs labelled by S.

Definition 1.3.8 *Let $\mathcal{P}$ be as in* (1.4)*, and let n be a non-negative integer. Then $\mathcal{P}$ is said to be n-***Cockcroft** *if $exp_S(\mathbb{P}) \equiv 0 \ (mod\ n)$, (where congruence $(mod\ 0)$ is taken to be equality) for all $S \in \mathbf{s}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. A monoid M is said to be n-Cockcroft if it admits an n-Cockcroft presentation.*

Remark 1.3.9 *To verify that the n-Cockcroft property holds, it is enough to check for pictures $\mathbb{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a trivialiser of $\mathcal{D}(\mathcal{P})$.*

The 0-Cockcroft property is usually just called Cockcroft.

In practice, we usually take n to be 0 or a prime p.

Example 1.3.2 (continued) *By* [60]*, trivialiser of $\mathcal{D}(\mathcal{P})$ contains the single picture $\mathbb{P}$ depicted in* 1.2.(i)*. Since* $\exp_R(\mathbb{P}) = \exp_S(\mathbb{P}) = \exp_T(\mathbb{P}) = 1 - 1 = 0$*, then $\mathcal{P}$ is*

Cockcroft. $\diamondsuit$

Example 1.3.5 (continued) *Since* $\exp_R(\mathbb{P}) = 1 - 3 = -2$, $\exp_S(\mathbb{P}) = 2 - 2 = 0$, $\exp_T(\mathbb{P}) = 1 - 1 = 0$ *then* $\mathcal{P}$ *is* 2*-Cockcroft.* $\diamondsuit$

Note that

$$\text{Aspherical} \Rightarrow \text{CA} \Rightarrow \text{Cockcroft} \Rightarrow n\text{-Cockcroft} \quad (n \in \mathbb{Z}^+).$$

1.3.4 Efficiency of monoid presentations

Let M be a monoid with the presentation $\mathcal{P}$, as in (1.4). As with group presentations we define the *Euler characteristic* of $\mathcal{P}$ by

$$\chi(\mathcal{P}) = 1 - |\mathbf{y}| + |\mathbf{s}|.$$

Let

$$\delta(M) = 1 - rk_{\mathbb{Z}}(H_1(M)) + d(H_2(M)),$$

where $rk_{\mathbb{Z}}(\ )$ denotes the $\mathbb{Z}$-rank of the torsion-free part and $d(\ )$ means the minimal number of generators. Then we have

Theorem 1.3.10 (Pride-unpublished)

$$\chi(\mathcal{P}) \geqslant \delta(M).$$

Then we define

$$\chi(M) = min\{\chi(\mathcal{P}) : \mathcal{P} \text{ a finite presentation for } M\}.$$

We should remark that some authors consider, just as with the group presentations,

$$-|\mathbf{y}| + |\mathbf{s}|,$$

and call this the *deficiency* of the presentation $\mathcal{P}$, denote by $def(\mathcal{P})$. The deficiency of a monoid M, denote by $def(M)$, is then taken to be the minimum deficiencies of all finite presentations of M. Clearly

$$1 + def(\mathcal{P}) = \chi(\mathcal{P}),$$

$$1 + def(M) = \chi(M).$$

Definition 1.3.11 *Let M be a monoid.*

i) A presentation $\mathcal{P}_0$ for M is called **minimal** *if*

$$\chi(\mathcal{P}_0) \leqslant \chi(\mathcal{P}),$$

for all presentations $\mathcal{P}$ of M.

ii) A finite presentation $\mathcal{P}$ is called **efficient** *if*

$$\chi(\mathcal{P}) = \delta(M).$$

iii) M is called **efficient** *if*

$$\chi(M) = \delta(M).$$

Theorem 1.3.12 (Pride-unpublished) *Let $\mathcal{P}$ be as in* (1.4)*. Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

As a consequence of the above theorem, we have

Corollary 1.3.13 *Let $\mathcal{P}$ be as in* (1.4)*. If $\mathcal{P}$ is Cockcroft then $\mathcal{P}$ is efficient.*

Let ψ be a ring homomorphism from $\mathbb{Z}M$ into the ring of all $k \times k$ martices over a comutative ring A with 1, for some $k \geqslant 1$, and suppose $\psi(1) = I_{k\times k}$.

Theorem 1.3.14 (Pride-unpublished) *Let $\mathbf{Y}$ be a trivializer of $\mathcal{D}(\mathcal{P})$. If*

either (a) $\psi(\lambda_{\mathbb{P},S}) = 0$ for all $\mathbb{P} \in \mathbf{Y}$, $S \in \mathbf{s}$,

or (b) $\psi(\eta_{\mathbb{P},S}) = 0$ for all $\mathbb{P} \in \mathbf{Y}$, $S \in \mathbf{s}$,

then $\mathcal{P}$ is minimal.

The above theorem can be restated as follows.

Theorem 1.3.15 *If there is a ring homomorphism ψ as above such that either $I_2^{(l)}(\mathcal{P})$ or $I_2^{(r)}(\mathcal{P})$ is contained in the kernel of ψ, then $\mathcal{P}$ is minimal.*

One of our main results (see Theorem 5.3.1) concern *minimal* but *inefficient* monoid presentations.

Some other examples of efficient and inefficient monoid presentation can be found, for example, in [2].

Chapter 2

The p-Cockcroft property of central extensions of groups

2.1 Introduction

A presentation for an arbitrary group extension is well-known, see for instance [6]. Also a generalization of the work in [19] on central extensions is presented in [6]. As an application of this we discuss necessary and sufficient conditions for the presentation of the central extension to be p-Cockcroft, where p is a prime or 0. Finally, we present some examples of this result.

2.2 Central extensions

Let Q be a group with the presentation $\mathcal{P}_Q = \langle \mathbf{a} \ ; \ \mathbf{r} \rangle$, and let K be a cyclic group of order m generated by x ($m = 0$ if x has infinite order). Any *central extension* of K by Q will have a presentation of the form

$$\mathcal{P} = \left\langle \mathbf{a}, \, x \ ; \ Rx^{-k_R} \, (R \in \mathbf{r}), \, x^m, \, [a, x] \, (a \in \mathbf{a}) \right\rangle, \tag{2.1}$$

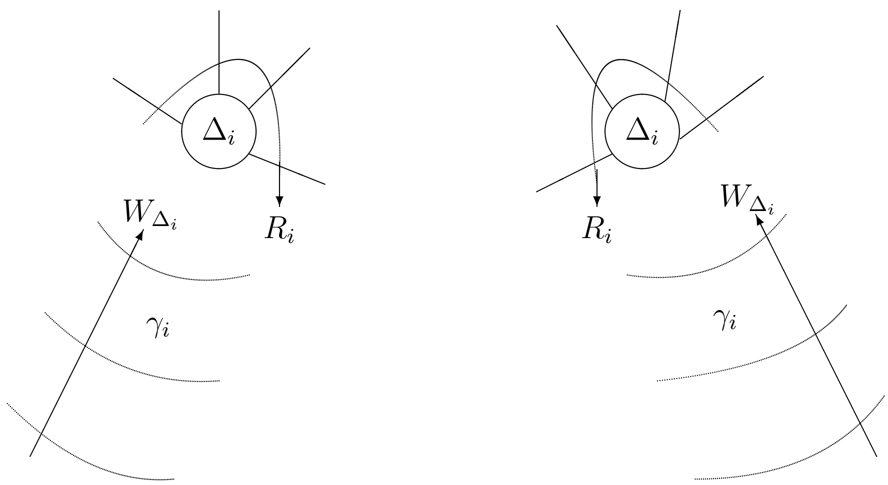
where $0 \leqslant k_R < m$, ($k_R \in \mathbb{Z}$ if $m = 0$).

However, not every presentation of this form defines an extension of K by Q because the order of x may not be m in $G \cong G(\mathcal{P})$. But, by [19] (see also [6, Corollary 7.2]), if we know a generating set, say $\mathbf{Y}$, of $\pi_2(\mathcal{P}_Q)$ then we can give necessary and sufficient conditions for x to have order m (Theorem 2.2.1 below).

Let $\mathbb{Q}$ ($\mathbb{Q} \in \mathbf{Y}$) have discs $\Delta_1, \Delta_2, \cdots, \Delta_t$ labelled $R_1^{\varepsilon_1}, R_2^{\varepsilon_2}, \cdots, R_t^{\varepsilon_t}$ respectively ($R_i \in \mathbf{r}$, $\varepsilon_i = \pm 1$, $1 \leqslant i \leqslant t$). Then let us choose a spray

$$\gamma_1, \gamma_2, \cdots, \gamma_t \tag{2.2}$$

for $\mathbb{Q}$, and suppose the label on γ_i is W_{Δ_i} which is a word on $\mathbf{a}$ ($1 \leqslant i \leqslant t$). This can be illustrated as in the following figure.



($\varepsilon_i = 1$) ($\varepsilon_i = -1$)

Let

$$\beta(\mathbb{Q}) = \sum_{i=1}^{t} \varepsilon_i k_{R_i}.$$

Theorem 2.2.1 ([6], [19]) *Let G be the group defined by the presentation* (2.1)*. Then the order of x is m in G if and only if*

$$\beta(\mathbb{Q}) \equiv 0 \ (mod\ m) \quad (\mathbb{Q} \in \mathbf{Y}). \tag{2.3}$$

For $\mathbb{Q} \in \mathbf{Y}$ as above and $a \in \mathbf{a}$, we let

$$\alpha_a(\mathbb{Q}) = \sum_{i=1}^{t} \varepsilon_i \exp_a(W_{\Delta_i}) k_{R_i}.$$

2.3 The p-Cockcroft property for the central extensions

2.3.1 The general theorem

Theorem 2.3.1 *Let p be a prime or 0, and let $\mathcal{P}$ be a presentation as in (2.1) such that the condition (2.3) holds. Then $\mathcal{P}$ is p-Cockcroft if and only if*

(i) $m \equiv 0\,(mod\ p)$,

(ii) $\exp_a(R) \equiv 0\,(mod\ p)$, *for all* $a \in \mathbf{a}$, $R \in \mathbf{r}$,

(iii) $\mathcal{P}_Q$ *is p-Cockcroft,*

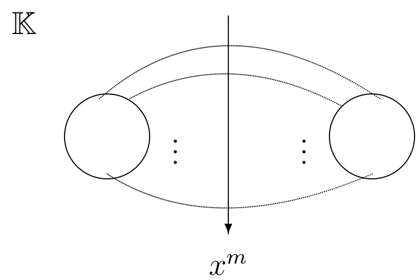
(iv) $\alpha_a(\mathbb{Q}) \equiv 0\,(mod\ p)$, *for all* $a \in \mathbf{a}$, $\mathbb{Q} \in \mathbf{Y}$,

(v) $\beta(\mathbb{Q}) \equiv 0\,(mod\ m.p)$, *for all* $\mathbb{Q} \in \mathbf{Y}$.

2.3.2 The generating pictures of $\pi_2(\mathcal{P})$

Let $\mathcal{P}$ be as in (2.1) such that the condition (2.3) holds. Now, by using [6], we can give a set of generating pictures over $\mathcal{P}$ as follows.

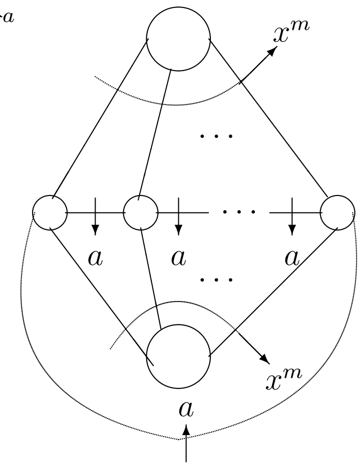
(I) The generating picture of the presentation $\mathcal{P}_K = \langle x\ ;\ x^m \rangle$ which is illustrated by



Note that if $m = 0$ then the above picture simply becomes the empty picture.

(II) For each $a \in \mathbf{a}$, we have a spherical picture

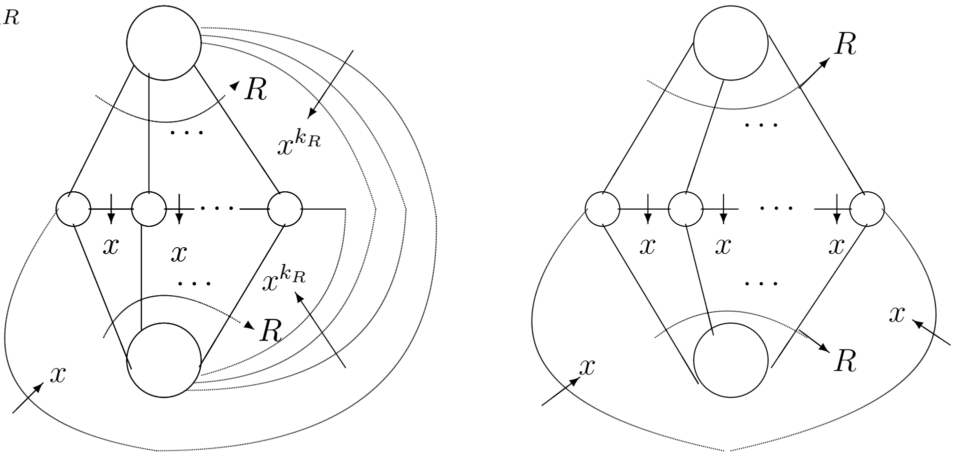
$\mathbb{K}_a$



Note also that if $m = 0$ then again the above picture becomes the empty picture.

(III) For each $R \in \mathbf{r}$, we have a spherical picture as in (**a**) (or (**b**) if $k_R = 0$) below.

$\mathbb{D}_{x,R}$



(**a**) (**b**)

(IV) For each $\mathbb{Q} \in \mathbf{Y}$, a picture $\hat{\mathbb{Q}}$ defined as follows.

For the picture $\mathbb{Q}$, we have the spray (2.2). Then, for each disc Δ_i labelled $R_i^{\varepsilon_i}$ ($1 \leqslant i \leqslant t$), we replace the spray line (transverse path) γ_i by a picture consisting of discs labelled $[a, x]$ ($a \in \mathbf{a}$) and with boundary label $W_{\Delta_i} x^{\varepsilon_i k_{R_i}} W_{\Delta_i}^{-1} x^{-\varepsilon_i k_{R_i}}$. This can be illustrated as in Figure 2.1. This gives a picture $\mathbb{Q}^*$ with the boundary label

$$\begin{aligned} W(\mathbb{Q}) &= (x^{\varepsilon_1 k_{R_1}} x^{\varepsilon_2 k_{R_2}} \cdots x^{\varepsilon_t k_{R_t}})^{-1} \\ &= x^{-\beta(\mathbb{Q})} \text{ by the definition of } \beta(\mathbb{Q}). \end{aligned}$$

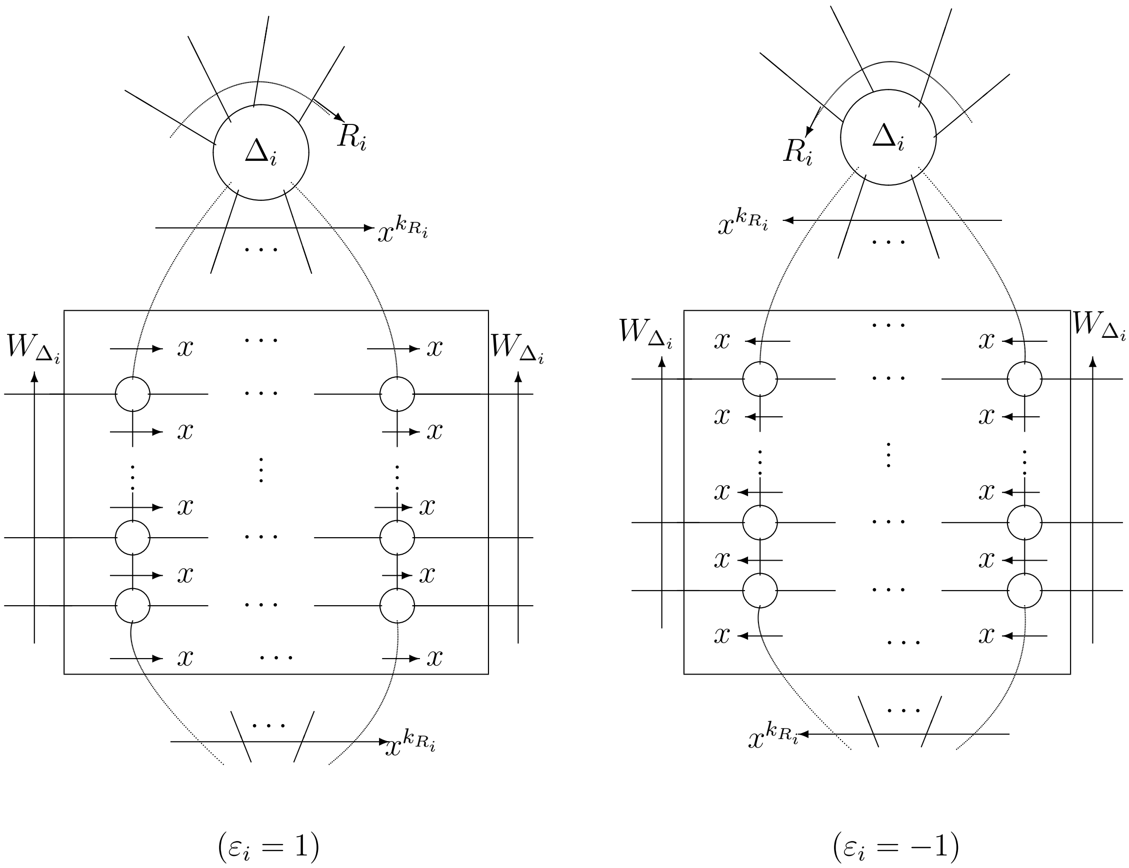
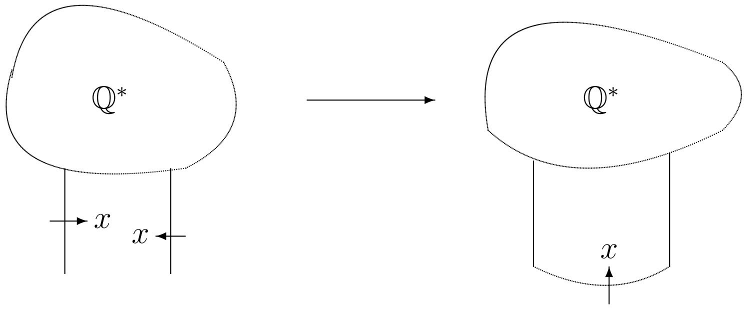


Figure 2.1:

We then cap off $\mathbb{Q}^*$ with a picture consisting of $\dfrac{\beta(\mathbb{Q})}{m}$ times x^m-discs (where $\dfrac{\beta(\mathbb{Q})}{m}$ is taken to be 0 if $m = 0$), positively oriented if $\beta(\mathbb{Q}) > 0$, negatively oriented if $\beta(\mathbb{Q}) < 0$, to obtain a spherical picture $\hat{\mathbb{Q}}$. In doing this it may be necessary to join up loose oppositely oriented x-arcs. This can be illustrated as in the following figure (see also Example 2.3.2 below).



Example 2.3.2 *Let Q be the group defined by the presentation*

$$\mathcal{P}_Q = \langle a,\, b\,;\; a^3,\, aba^{-1}b^{-1} \rangle\,,$$

and let K be the cyclic group of order 3 generated by x. Consider the presentation

$$\mathcal{P} = \langle a,\, b,\, x\,;\; a^3x^{-1},\, aba^{-1}b^{-1}x^{-2},\, x^3,\, [a,x],\, [b,x] \rangle\,.$$

By [3], $\pi_2(\mathcal{P}_Q)$ *is generated by the pictures* $\mathbb{Q}_1$ *and* $\mathbb{Q}_2$ *depicted in Figure* 2.2. *We*

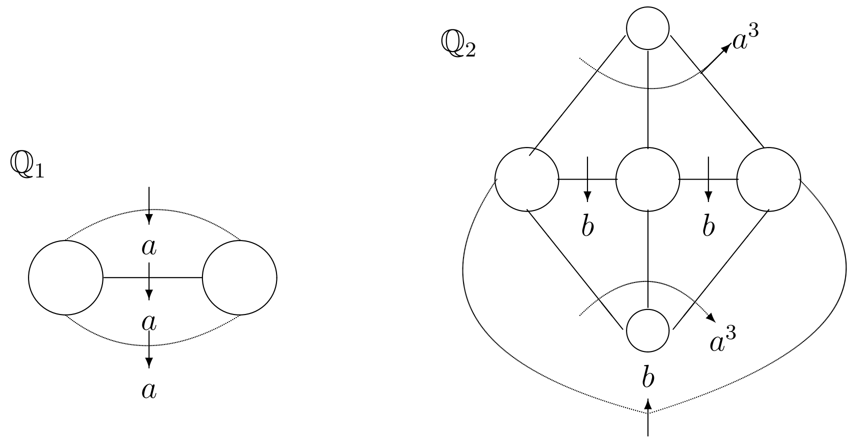


Figure 2.2:

have $\beta(\mathbb{Q}_1) = 0$, $\beta(\mathbb{Q}_2) = 6$. *So* (2.3) *holds. Hence, by Theorem* 2.2.1, *the group G defined by $\mathcal{P}$ is a central extension of K by Q. We get the pictures* $\mathbb{Q}_1^*$, $\mathbb{Q}_2^*$ *as in Figure* 2.3. *Then we obtain* $\hat{\mathbb{Q}}_1$, $\hat{\mathbb{Q}}_2$ *as in Figure* 2.4. $\diamondsuit$

The proof of the following theorem can be found in [6, Theorem 6.4].

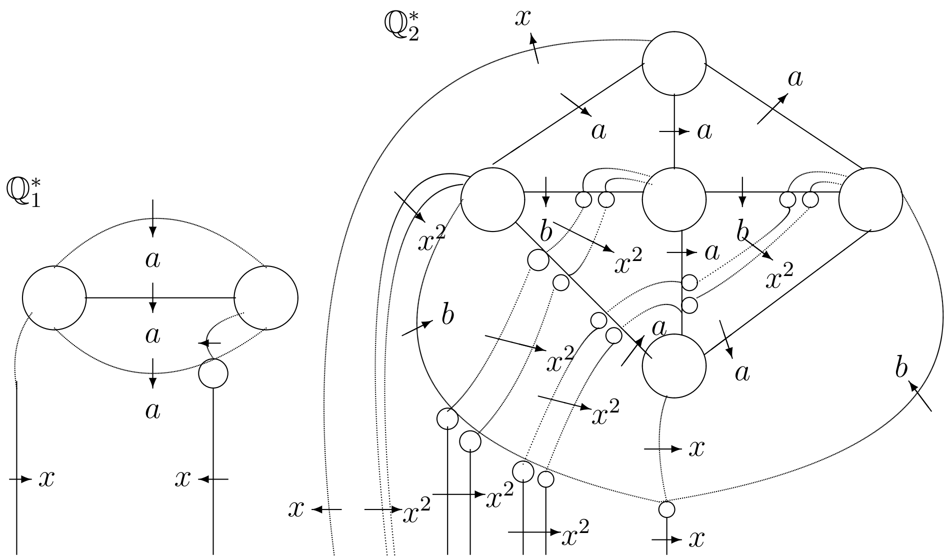


Figure 2.3:

Theorem 2.3.3 *Let $\mathcal{P}$ be as in* (2.1) *such that the condition* (2.3) *holds. Then* $\pi_2(\mathcal{P})$ *is generated by the pictures*

$$\mathbb{K},\ \mathbb{K}_a\ (a \in \mathbf{a}),\ \mathbb{D}_{x,R}\ (R \in \mathbf{r})\ \text{ and }\ \hat{\mathbb{Q}}\ (\mathbb{Q} \in \mathbf{Y}).$$

2.3.3 The proof of Theorem 2.3.1

Let C_R, C_a denote the relators Rx^{-k_R} $(R \in \mathbf{r})$, $[a, x]$ $(a \in \mathbf{a})$ respectively.

First assume that $m \neq 0$.

Let us consider the picture $\mathbb{K}$. It is clear that $\exp_{x^m}(\mathbb{K}) = 1 - 1 = 0$. Also, let us consider a picture $\mathbb{K}_a$ $(a \in \mathbf{a})$. Clearly $\exp_{x^m}(\mathbb{K}_a) = 1 - 1 = 0$, and it is easy to see that

$$\exp_{C_a}(\mathbb{K}_a) = \exp_x(x^m) = m,$$

so we must have $m \equiv 0 \pmod{p}$. Hence the condition (i) must hold.

Consider a picture $\mathbb{D}_{x,R}$ $(R \in \mathbf{r})$. It is clear that

$$\exp_{C_R}(\mathbb{D}_{x,R}) = 1 - 1 = 0.$$

Also it is easy to see that

$$\exp_{C_a}(\mathbb{D}_{x,R}) = \exp_a(R),$$

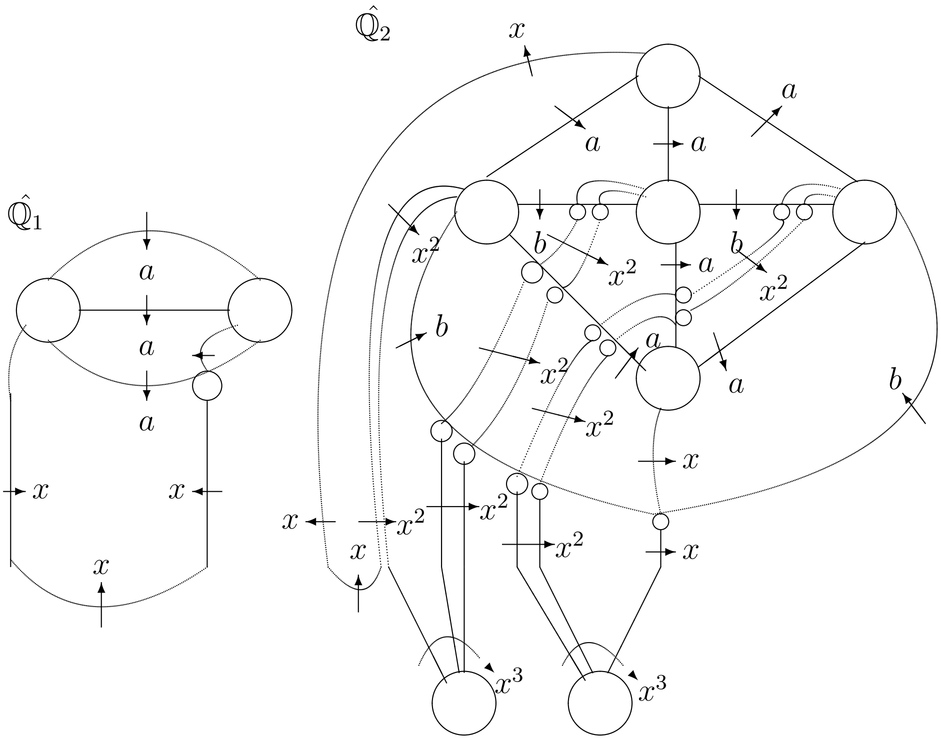


Figure 2.4:

for all $a \in \mathbf{a}$. Thus the condition (ii) must hold.

Now consider a picture $\hat{\mathbb{Q}}$ ($\mathbb{Q} \in \mathbf{Y}$). We must have $\exp_{C_R}(\hat{\mathbb{Q}}) \equiv 0 \pmod{p}$. But

$$\exp_{C_R}(\hat{\mathbb{Q}}) = \exp_R(\mathbb{Q}),$$

so we must have $\exp_R(\mathbb{Q}) \equiv 0 \pmod{p}$, that is, $\mathcal{P}_Q$ must be p-Cockcroft. This gives the condition (iii). Also, for a fixed $a \in \mathbf{a}$, it is easy to see that

$$\exp_{C_a}(\hat{\mathbb{Q}}) = \alpha_a(\mathbb{Q}).$$

So we must have $\alpha_a(\mathbb{Q}) \equiv 0 \pmod{p}$, which gives the condition (iv). Finally, we have that

$$\exp_{x^m}(\hat{\mathbb{Q}}) = \frac{\beta(\mathbb{Q})}{m}.$$

Then we must have $\beta(\mathbb{Q}) \equiv 0 \pmod{m.p}$. So the condition (v) must hold.

Suppose that $m = 0$.

Then the five conditions (i)-(v) reduce to the three conditions

(ii) $\exp_a(R) \equiv 0 \pmod p$, for all $a \in \mathbf{a}$, $R \in \mathbf{r}$,

(iii) $\mathcal{P}_Q$ is p-Cockcroft,

(iv) $\alpha_a(\mathbb{Q}) \equiv 0 \pmod p$, for all $a \in \mathbf{a}$, $\mathbb{Q} \in \mathbf{Y}$,

since the conditions (i) and (v) automatically hold. Because the pictures $\mathbb{K}$ and $\mathbb{K}_a$ are trivial, so impose no restrictions, and there are no x^m discs, then the above proof will carry over.

2.4 Some examples

Example 2.4.1 *Let Q be the $(k,\, l,\, n)$-triangle group with the presentation*

$$\mathcal{P}_Q = \left\langle a,\, b\; ;\; a^k,\, b^l,\, (ab)^n \right\rangle,$$

where $k,\, l,\, n \in \mathbb{Z}^+$ and

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{n} \leqslant 1,$$

and let K be a cyclic group of order m generated by x (m is taken to be 0 if x has infinite order). Consider the presentation

$$\mathcal{P} = \left\langle a,\, b,\, x\; ;\; a^k x^{-r},\, b^l x^{-s},\, (ab)^n x^{-t},\, x^m,\, C_a,\, C_b \right\rangle, \tag{2.4}$$

where $0 \leqslant r,\, s,\, t < m$ (or $r,\, s,\, t \in \mathbb{Z}$, if $m = 0$) and, as in the proof of Theorem 2.3.1,

$$C_a := [a, x] \quad \text{and} \quad C_b := [b, x].$$

By the weight test (see [11], [24]), $\mathcal{P}_Q$ is CA (and then Cockcroft). We can give a set of generating pictures of $\pi_2(\mathcal{P}_Q)$, as in Figure 2.5. We have $\beta(\mathbb{Q}_1) = 0$, $\beta(\mathbb{Q}_2) = 0$ and $\beta(\mathbb{Q}_3) = 0$. So (2.3) holds. Hence, by Theorem 2.2.1, the group G defined by $\mathcal{P}$ is a central extension of K by Q.

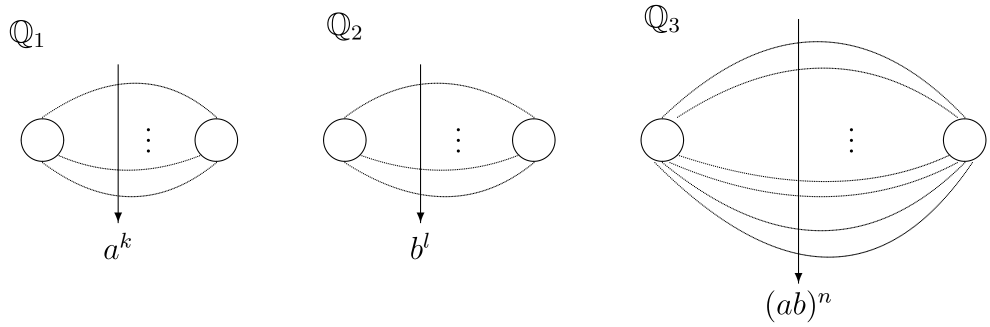


Figure 2.5:

We also have

$$\exp_a(a^k) = k, \qquad \exp_b(b^l) = l,$$
$$\exp_a((ab)^n) = n, \qquad \exp_b((ab)^n) = n.$$

Moreover, by the definition, we get

$$\alpha_a(\mathbb{Q}_1) = r, \qquad \alpha_b(\mathbb{Q}_1) = 0,$$
$$\alpha_a(\mathbb{Q}_2) = 0, \qquad \alpha_b(\mathbb{Q}_2) = s,$$
$$\alpha_a(\mathbb{Q}_3) = t, \qquad \alpha_b(\mathbb{Q}_3) = t.$$

Also, for any prime p, we always have

$$\beta(\mathbb{Q}_i) \equiv 0 \, (mod \; m.p) \qquad (i = 1, 2, 3).$$

$\Diamond$

Thus, we get the following result for the above example, as a consequence of Theorems 2.3.1 and 1.2.15.

Corollary 2.4.2 *Let p be a prime. Then the presentation $\mathcal{P}$, as in* (2.4), *is p-Cockcroft if and only if*

$$m \equiv 0 \, (mod \; p),$$
$$k \equiv 0 \, (mod \; p), \quad l \equiv 0 \, (mod \; p), \qquad n \equiv 0 \, (mod \; p),$$
$$r \equiv 0 \, (mod \; p), \quad s \equiv 0 \, (mod \; p), \qquad t \equiv 0 \, (mod \; p).$$

Hence $\mathcal{P}$ is efficient if and only if

$$hcf(m,\, k,\, l,\, n,\, r,\, s,\, t) \neq 1.$$

Example 2.4.3 *Let Q be the group $\mathbb{Z}_k \oplus \mathbb{Z}_l$ $(k, l \in \mathbb{Z}^+)$ with the presentation*

$$\mathcal{P}_Q = \left\langle a,\, b\,;\; a^k,\, b^l,\, [a,b] \right\rangle,$$

and let K be a finite cyclic group of order m generated by x. Let us consider the presentation

$$\mathcal{P} = \left\langle a,\, b,\, x\,;\; a^k x^{-r},\, b^l x^{-s},\, [a,b]x^{-t},\, x^m,\, C_a,\, C_b \right\rangle, \tag{2.5}$$

where $0 \leqslant r, s, t < m$ and again, as in the proof of Theorem 2.3.1,

$$C_a := [a,x] \quad \textit{and} \quad C_b := [b,x].$$

We can give a set of generating pictures of $\pi_2(\mathcal{P}_Q)$, as in Figure 2.6. We have $\beta(\mathbb{Q}_1) =$

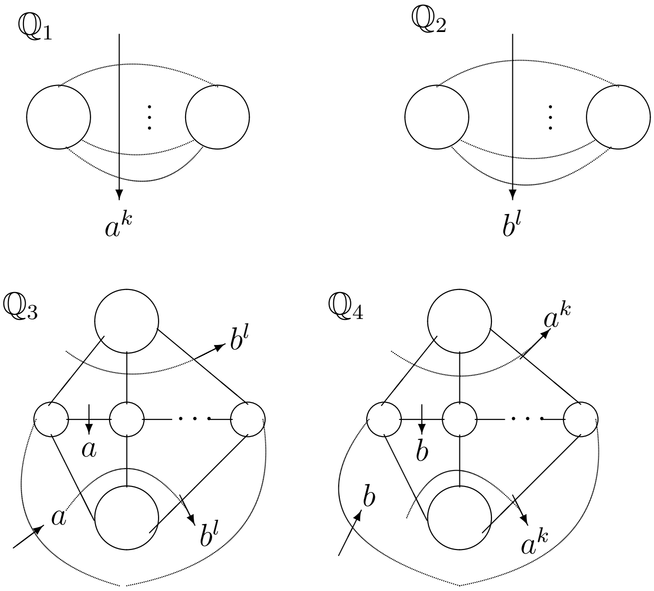


Figure 2.6:

0, $\beta(\mathbb{Q}_2) = 0$, $\beta(\mathbb{Q}_3) = lt$ and $\beta(\mathbb{Q}_4) = kt$.

Suppose that

$$lt \equiv 0\;(mod\; m) \quad \textit{and} \quad kt \equiv 0\;(mod\; m).$$

So (2.3) *holds. Then, by Theorem* 2.2.1*, the group G defined by $\mathcal{P}$ is a central extension of K by Q.*

It is clear that

$$\exp_a(a^k) = k, \qquad \exp_b(b^l) = l,$$
$$\exp_a([a,b]) = 1 - 1 = 0 \qquad \exp_b([a,b]) = 1 - 1 = 0.$$

Also, by the definition, we get

$$\alpha_a(\mathbb{Q}_1) = r, \qquad \alpha_b(\mathbb{Q}_1) = 0,$$
$$\alpha_a(\mathbb{Q}_2) = 0, \qquad \alpha_b(\mathbb{Q}_2) = s,$$
$$\alpha_a(\mathbb{Q}_3) = s, \qquad \alpha_b(\mathbb{Q}_3) = -\frac{1}{2}l(l-1)t,$$
$$\alpha_a(\mathbb{Q}_4) = \frac{1}{2}k(k-1)t, \qquad \alpha_b(\mathbb{Q}_3) = r.$$

$\lozenge$

Therefore, we get the following result for the above example, as a consequence of Theorems 2.3.1 and 1.2.15.

Corollary 2.4.4 *Let p be a prime. Then the presentation $\mathcal{P}$, as in* (2.5)*, is p-Cockcroft if and only if*

$$m \equiv 0 \,(mod\ p),$$
$$k \equiv 0 \,(mod\ p), \quad r \equiv 0 \,(mod\ p), \quad kt \equiv 0 \,(mod\ m.p), \quad \tfrac{1}{2}k(k-1)t \equiv 0 \,(mod\ p),$$
$$l \equiv 0 \,(mod\ p), \quad s \equiv 0 \,(mod\ p), \quad lt \equiv 0 \,(mod\ m.p), \quad -\tfrac{1}{2}l(l-1)t \equiv 0 \,(mod\ p).$$

Thus $\mathcal{P}$ is efficient if and only if

$$hcf(m,k,l,r,s,\frac{1}{2}k(k-1)t,\frac{1}{2}l(l-1)t,\frac{1}{m}kt,\frac{1}{m}lt) \neq 1.$$

Chapter 3

The efficiency of standard wreath products of groups

3.1 Some background

Let ξ_p denote the set of all finite p-groups (p a prime) which have efficient presentations. In 1970, Johnson [36] showed that ξ_p is closed under direct products and after that, for p odd, ξ_p is closed under standard wreath products. Also in 1973, Wamsley [58] showed that ξ_p is closed under general wreath products.

Let ξ be the set of all finite groups which have efficient presentations. In this chapter we will give sufficient conditions for the standard wreath product of any two groups which belong to ξ to be efficient.

Definition 3.1.1 *If there are given*

a-) a group A,

b-) a group K,

c-) a homomorphisim θ of A into the automorphisim group of K

$$\theta : A \longrightarrow Aut(K), \qquad a \longmapsto \theta_a$$

for all $a \in A$,

then the semi-direct product $G = K \rtimes_\theta A$ of K by A is defined as follows.

The elements of G are all ordered pairs (a, k) $(a \in A,\ k \in K)$ and multiplication is given by

$$(a, k)(a', k') = (aa', (k\theta_{a'})k').$$

Similiar definitions of a semi-direct product can be found in [7] or [54]. We remark that semi-direct products of **monoids** will be discussed (in detail) in Chapter 4, Section 4.3.

The proof of the following Lemma can be found in [35, Proposition 10.1, Corollary 10.1].

Lemma 3.1.2 *Suppose that $\mathcal{P}_K = \langle \mathbf{y}; \mathbf{s} \rangle$ and $\mathcal{P}_A = \langle \mathbf{x}; \mathbf{r} \rangle$ are presentations for the groups K and A respectively under the maps*

$$y \longmapsto k_y \ \ (y \in \mathbf{y}), \qquad x \longmapsto a_x \ \ (x \in \mathbf{x}).$$

Then we have a presentation for $G = K \rtimes_\theta A$

$$\mathcal{P} = \langle \mathbf{y},\, \mathbf{x}\,;\, \mathbf{s},\, \mathbf{r},\, \mathbf{t} \rangle$$

where $\mathbf{t} = \{yx\lambda_{yx}^{-1}x^{-1} | y \in \mathbf{y}, x \in \mathbf{x}\}$, and λ_{yx} is a word on $\mathbf{y}$ representing the element $(k_y)\theta_{a_x}$ of K $(a \in A,\ k \in K,\ x \in \mathbf{x},\ y \in \mathbf{y})$.

Now let us define the standard wreath product by using Definition 3.1.1.

Definition 3.1.3 *Let A and B be finite groups with $A = \{a_1, a_2, \cdots, a_g\}$, say. Let x be any element of A. Then,*

$$a_1x, a_2x, \cdots, a_gx$$

is a permutation of $a_1, a_2, \cdots, a_g$. So we can write $a_1x, a_2x, \cdots, a_gx$ as $a_{\sigma_x(1)}, a_{\sigma_x(2)}, \cdots, a_{\sigma_x(g)}$ where σ_x is a permutation of $1, 2, \cdots, g$.

Let K be the direct product of the number of $|A|$ copies of B, that is,

$$K = B^{|A|} = B^g = \underbrace{B \times B \times \cdots \times B}_{(g\, times)},$$

and let $(b_{a_1}, b_{a_2}, \cdots, b_{a_g})$ *be a typical element of* K*. We have a homomorphism*

$$\theta : A \longrightarrow Aut(K), \qquad x \longmapsto \theta_x$$

where

$$(b_{a_1}, b_{a_2}, \cdots, b_{a_g})\theta_x = (b_{a_{\sigma x(1)}}, b_{a_{\sigma x(2)}}, \cdots, b_{a_{\sigma x(g)}}).$$

The split extension $K \rtimes_\theta A$ *is called the standard wreath product of* B *by* A*, denoted* $B \wr A$*.*

We should note that some authors, for instance Karpilovsky in [38], use the notation $A \wr B$ instead of $B \wr A$. Here we use the notation as in [54].

We also need the following well known results.

Proposition 3.1.4 (Schur 1904) *Let* B *be a finite group. Then*

(i) $H_2(B)$ *is a finite group, whose elements have order dividing the order of* B*.*

(ii) $H_2(B) = 1$ *if* B *is cyclic.*

Definition 3.1.5 1) *Given an abelian group* A*, we denote by* $A\#A$ *the factor group of* $A \otimes A$ *by the subgroup generated by the elements of the form* $a \otimes b + b \otimes a$*,* $(a, b \in A)$*.*

2) *In any group* K*, an element of order* 2 *is called an "involution".*

Theorem 3.1.6 (Blackburn 1972) *Let* m *denote the number of involutions in the group* A*. Then* $H_2(B \wr A)$ *is the direct sum of* $H_2(B)$*,* $H_2(A)$*,* $(1/2)(|A| - m - 1)$ *copies of* $H_1(B) \otimes H_1(B)$ *and* m *copies of* $H_1(B) \# H_1(B)$*.*

Lemma 3.1.7 *Let* B *be a finite group, let*

$$H_1(B) \cong \bigoplus_{i=1}^{t} \mathbb{Z}_{n_i},$$

and let s be the number of even n_i, $1 \leqslant i \leqslant t$. Then,

$$H_1(B) \# H_1(B) \;\cong\; \bigoplus_{1 \leqslant i < j \leqslant t} \mathbb{Z}_{(n_i, n_j)} \oplus \mathbb{Z}_2^{(s)},$$

where $\mathbb{Z}_2^{(s)}$ is a direct sum of s copies of $\mathbb{Z}_2$.

Proofs of Proposition 3.1.4, Theorem 3.1.6 and Lemma 3.1.7 can be found in [38].

Lemma 3.1.8 (Kunneth Formula) *Let A and B be any two groups and let $G = A \times B$. Then,*

$$H_2(G) = H_2(A) \oplus H_2(B) \oplus H_1(A) \otimes H_1(B).$$

Definition 3.1.9 *Let A be a finite abelian group. Then we define,*

$$t(A) = \begin{cases} first\, torsion\, number & if \quad A \neq 0 \\ 0 & if \quad A = 0 \end{cases}.$$

Lemma 3.1.10 *Let A and B be finite abelian groups. If $(t(A)\,,\, t(B)) \neq 1$ then*

$$d(A \oplus B) = d(A) + d(B).$$

Proof. Suppose that $(t(A)\,,\, t(B)) \neq 1$.

First of all, if one of $t(A)$ or $t(B)$ is 0, say $t(A)$ then by Definition 3.1.9, $A = 0$. Then, basically we have that $d(B) = d(B)$.

Now suppose both $t(A)$ and $t(B)$ are non-zero, and let

$$A = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k},$$

where $m_i \mid m_{i+1}$, $1 \leqslant i \leqslant k-1$. Then $t(A) = m_1$. Similarly, let

$$B = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_l},$$

where $n_j \mid n_{j+1}$, $1 \leqslant j \leqslant l-1$ and $t(B) = n_1$. Then,

$$A \oplus B = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_l}.$$

Now, let p be a prime with $p \mid t(A)$ and $p \mid t(B)$. Then

$$p \mid m_1, p \mid m_2, \cdots, p \mid m_k, p \mid n_1, p \mid n_2, \cdots, p \mid n_l.$$

So there are epimorphisms

$$\phi_i : \mathbb{Z}_{m_i} \twoheadrightarrow \mathbb{Z}_p \text{ and } \psi_j : \mathbb{Z}_{n_j} \twoheadrightarrow \mathbb{Z}_p,$$

where $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$. Then we get induced epimorphisms

$$\phi = \bigoplus_{1 \leqslant i \leqslant k} \phi_i : \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k} \twoheadrightarrow \mathbb{Z}_p^{(k)}$$

and

$$\psi = \bigoplus_{1 \leqslant j \leqslant l} \psi_j : \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_l} \twoheadrightarrow \mathbb{Z}_p^{(l)}.$$

Then

$$\phi \oplus \psi : \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_l} \twoheadrightarrow \mathbb{Z}_p^{(k+l)}.$$

Now since $\mathbb{Z}_p^{(k)}$ is a vector space over $\mathbb{Z}_p$ [54, Lemma 6.2], and since any two bases of a vector space have the same cardinality [33, Theorem 4.2.7], that is, the dimension of $\mathbb{Z}_p^{(k)}$, then we have that $\mathbb{Z}_p^{(k)}$ cannot be generated by less than k elements. In other words, $d(\mathbb{Z}_p^{(k)}) = k$. Thus, by the fact that the minimal number of generators of a group is greater than or equal to the minimal number of generators of any homomorphic image of that group, we have that

$$d(A) \geqslant k.$$

On the other hand, A can be generated by k elements which are

$$(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1).$$

So, $d(A) \leqslant k$, then $d(A) = k$. Similarly, $d(B) = l$ and $d(A \oplus B) = k + l$. □

Remark 3.1.11 *Clearly we can generalize this lemma for more than two abelian groups, that is, if* A_i $(1 \leqslant i \leqslant n)$ *are abelian groups and* $(t(A_1), t(A_2), \cdots, t(A_n)) \neq 1$ *then*

$$d(A_1 \oplus A_2 \oplus \cdots \oplus A_n) = d(A_1) + d(A_2) + \cdots + d(A_n).$$

3.2 The main theorem

Throught this section A, B will be finite groups satisfying the following conditions.

(i) A, B have efficient presentations $\mathcal{P}_A = \langle \mathbf{x} \ ; \ \mathbf{r} \rangle$ and $\mathcal{P}_B = \langle \mathbf{y} \ ; \ \mathbf{s} \rangle$ respectively on g, n $(g, n \in \mathbb{N})$ generators where $n = d(B)$,

(ii) $d(B) = d(H_1(B))$,

(iii) **either** the orders of A, $H_1(B)$ are even and also $t(H_2(A))$, $t(H_2(B))$ and $t(H_1(B))$ are even **or** the order of A is odd and there exists an odd prime p dividing $t(H_2(A))$,
$t(H_2(B))$ and $t(H_1(B))$, where t is defined as in Definition 3.1.9.

Theorem 3.2.1 (Main Theorem) *Let $G = B \wr A$. Then G has an efficient presentation.*

The proof of the following remark can be found at the end of this section as a lemma.

Remark 3.2.2 *Suppose $g = d(A)$. If $(t(H_1(A)), t(H_1(B))) \neq 1$ and $d(H_1(A)) = d(A)$ then $d(G) = g + n$.*

The proof of Theorem 3.2.1 will proceed by the following steps.

3.2.1 Calculation of $\delta(B \wr A)$ and $d(H_2(B \wr A))$

In this part of the proof, we will calculate

$$\delta(G) = 1 + rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G)).$$

Now since G is a finite group then $rk_{\mathbb{Z}}(H_1(G))$ is trivial, so we will just calculate

$$\delta(G) = 1 + d(H_2(G)).$$

Recall that we had a formula to calculate $H_2(G)$ by Theorem 3.1.6, that is,

$$\begin{aligned} H_2(G) &= H_2(B \wr A) \\ &= H_2(B) \oplus H_2(A) \oplus (H_1(B) \otimes H_1(B))^{\frac{1}{2}(|A|-m-1)} \oplus (H_1(B) \# H_1(B))^m, \end{aligned} \tag{3.1}$$

where m denotes the number of involutions in the group A.

Let us write

$$H_1(B) = \mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2} \oplus \cdots \oplus \mathbb{Z}_{v_n}, \quad v_i \mid v_{i+1}, \quad 1 \leqslant i \leqslant n-1. \tag{3.2}$$

By (ii), $d(H_1(B)) = d(B) = n$.

At first, let us calculate the "$\otimes$" part in (3.1).

$$\begin{aligned} H_1(B) \otimes H_1(B) \;\; = \;\; & (\mathbb{Z}_{v_1} \otimes \mathbb{Z}_{v_1}) \oplus (\mathbb{Z}_{v_1} \otimes \mathbb{Z}_{v_2}) \oplus \cdots\cdots \oplus (\mathbb{Z}_{v_1} \otimes \mathbb{Z}_{v_n}) \oplus \\ & (\mathbb{Z}_{v_2} \otimes \mathbb{Z}_{v_1}) \oplus (\mathbb{Z}_{v_2} \otimes \mathbb{Z}_{v_2}) \oplus \cdots\cdots \oplus (\mathbb{Z}_{v_2} \otimes \mathbb{Z}_{v_n}) \oplus \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & (\mathbb{Z}_{v_n} \otimes \mathbb{Z}_{v_1}) \oplus (\mathbb{Z}_{v_n} \otimes \mathbb{Z}_{v_2}) \oplus \cdots\cdots \oplus (\mathbb{Z}_{v_n} \otimes \mathbb{Z}_{v_n}) \oplus \\ = \;\; & \mathbb{Z}_{v_1} \oplus \mathbb{Z}_{(v_1,v_2)} \oplus \cdots\cdots \oplus \mathbb{Z}_{(v_1,v_n)} \oplus \\ & \mathbb{Z}_{(v_2,v_1)} \oplus \mathbb{Z}_{v_2} \oplus \cdots\cdots \oplus \mathbb{Z}_{(v_2,v_n)} \oplus \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \mathbb{Z}_{(v_n,v_1)} \oplus \mathbb{Z}_{(v_n,v_2)} \oplus \cdots\cdots \oplus \mathbb{Z}_{v_n}. \end{aligned}$$

Since $v_1 \mid v_2 \mid \cdots \mid v_n$, we have

$$\begin{array}{ccccccccc} (v_1, v_2) & = & (v_1, v_3) & = & \cdots & = & (v_1, v_n) & = & v_1 \\ (v_2, v_3) & = & (v_2, v_4) & = & \cdots & = & (v_2, v_n) & = & v_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & \vdots & \vdots & \vdots \\ & & & & & & (v_{n-1}, v_n) & = & v_{n-1}. \end{array}$$

So the sum becomes

$$= \;\; \mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_1} \oplus \cdots \oplus \mathbb{Z}_{v_1} \oplus$$

$$\begin{aligned}
&\mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2} \oplus \cdots \oplus \mathbb{Z}_{v_2} \oplus \\
&\mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2} \oplus \mathbb{Z}_{v_3} \oplus \cdots \oplus \mathbb{Z}_{v_3} \oplus \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_2} \oplus \cdots \oplus \mathbb{Z}_{v_n}.
\end{aligned}$$

Then $t(H_1(B) \otimes H_1(B)) = v_1$. Hence by Lemma 3.1.10,

$$d(H_1(B) \otimes H_1(B))^{\frac{1}{2}(|A|-m-1)} = \frac{1}{2}(|A| - m - 1)d(B)^2. \tag{3.3}$$

<u>*Case 1 : |A| is even*</u>

In this case we must calculate the "#" part in (3.1), as well.

Suppose that $t(H_1(B))$ is even. Then it implies that each term in the decomposition (3.2) of $H_1(B)$ is even. Now let us use the formula which is given in Lemma 3.1.7. So,

$$\begin{aligned}
H_1(B)\#H_1(B) \;=\; &\mathbb{Z}_{(v_1,v_2)} \oplus \mathbb{Z}_{(v_1,v_3)} \oplus \cdots\cdots \oplus \mathbb{Z}_{(v_1,v_n)} \oplus \\
&\mathbb{Z}_{(v_2,v_3)} \oplus \mathbb{Z}_{(v_2,v_4)} \oplus \cdots\cdots \oplus \mathbb{Z}_{(v_2,v_n)} \oplus \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\mathbb{Z}_{(v_{n-2},v_{n-1})} \oplus \mathbb{Z}_{(v_{n-2},v_n)} \oplus \mathbb{Z}_{(v_{n-1},v_n)} \oplus {\mathbb{Z}_2}^{(n)}.
\end{aligned}$$

(Since every term is even in $H_1(B)$ then we take n to be the power of $\mathbb{Z}_2$ in the above formula.) And by using the fact $v_1 \mid v_2 \mid \cdots \mid v_n$, the sum will become

$$\begin{aligned}
= \;&\mathbb{Z}_{v_1} \oplus \mathbb{Z}_{v_1} \oplus \cdots\cdots \oplus \mathbb{Z}_{v_1} \oplus \\
&\mathbb{Z}_{v_2} \oplus \mathbb{Z}_{v_2} \oplus \cdots\cdots \oplus \mathbb{Z}_{v_2} \oplus \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\mathbb{Z}_{v_{n-2}} \oplus \mathbb{Z}_{v_{n-2}} \oplus \mathbb{Z}_{v_{n-1}} \oplus {\mathbb{Z}_2}^{(n)}.
\end{aligned}$$

And so $t(H_1(B)\#H_1(B)) = 2$. Then by Lemma 3.1.10, we have that

$$d(H_1(B) \# H_1(B))^m \;=\; m(\frac{(n-1)[(n-1)+1]}{2\,!} + n)$$

$$
\begin{aligned}
&= m(\frac{n^2+n}{2}) \\
&= m(\frac{d(B)^2+d(B)}{2}).
\end{aligned}
$$

Therefore (again by Lemma 3.1.10, and using assumption (iii))

$$d(H_2(G)) = d(H_2(A)) + d(H_2(B)) + \frac{1}{2}(|A| - m - 1)\, d(B)^2 + \frac{1}{2}m\,(d(B)^2 + d(B)),$$

and after some rearrangements, we get

$$d(H_2(G)) = d(H_2(A)) + d(H_2(B)) + \frac{1}{2}\, d(B)^2(|A| - 1 + \frac{m}{d(B)}).$$

Therefore we have

$$\delta(G) = d(H_2(A)) + d(H_2(B)) + 1 + \frac{1}{2}\, d(B)^2(|A| - 1 + \frac{m}{d(B)}). \tag{3.4}$$

<u>*Case 2 : $|A|$ is odd*</u>

By assumption (iii), there exists an odd prime p such that

$$p \mid t(H_2(A))\ ,\ p \mid t(H_2(B))\ ,\ p \mid t(H_1(B)).$$

In this case, since the order of A is odd, we cannot have any involutions in group A, so the value m in the third and final terms of (3.1) becomes zero. Thus we will just need to calculate the "⊗" part in (3.1). Now by using Lemma 3.1.10, we have

$$d(H_1(B) \otimes H_1(B))^{\frac{1}{2}(|A|-1)} = \frac{1}{2}(|A| - 1)\, d(B)^2,$$

following the same calculation as in (3.3). Then by using assumption (iii) and Lemma 3.1.10, we get

$$d(H_2(G)) = d(H_2(A)) + d(H_2(B)) + \frac{1}{2}(|A| - 1)\, d(B)^2.$$

Therefore we have

$$\delta(G) = d(H_2(A)) + d(H_2(B)) + 1 + \frac{1}{2}\, d(B)^2(|A| - 1). \tag{3.5}$$

3.2.2 To obtain an efficient presentation for $G = B \wr A$

To get an efficient presentation for $G = B \wr A$, the following process can be followed:

- For each $a \in A$ take a copy $\left\langle \mathbf{y}^{(a)} \; ; \; \mathbf{s}^{(a)} \right\rangle$ of $\mathcal{P}_B$,
- Choose an ordering $a_1 < a_2 < \cdots < a_n$ of the elements of A where $a_1 = 1$,
- Let $\{a_x : x \in \mathbf{x}\}$ be a generating set for A corresponding to the presentation $\mathcal{P}_A = \langle \mathbf{x} \; ; \; \mathbf{r} \rangle$,
- Let $\{b_y : y \in \mathbf{y}\}$ be a generating set for B corresponding to the presentation $\mathcal{P}_B = \langle \mathbf{y} \; ; \; \mathbf{s} \rangle$.

Lemma 3.2.3 *A presentation of $G = B \wr A$ is given by*

$$\begin{aligned}\mathcal{P}_0 \;=\; & \Big\langle \mathbf{y}^{(a)} \; (a \in A), \mathbf{x} \; ; \; \mathbf{s}^{(a)} \;\; (a \in A), \mathbf{r}, \; y^{(a)} z^{(a')} = z^{(a')} y^{(a)} \;\; (a, a' \in A, \, a < a', \, y, z \in \mathbf{y}), \\ & x^{-1} y^{(a)} x = y^{(a a_x)} \;\; (a \in A, \, y \in \mathbf{y}, \, x \in \mathbf{x}) \Big\rangle .\end{aligned}$$

Proof. By Definition 3.1.3, K is the direct product of $|A|$ copies of B so that a presentation of K can be written

$$\mathcal{P}_K = \Big\langle \mathbf{y}^{(a)} \;\; (a \in A) \; ; \; \mathbf{s}^{(a)} \;\; (a \in A), \, [y^{(a)}, z^{(a')}] \;\; (a, a' \in A, \, a < a', \, y, z \in \mathbf{y}) \Big\rangle .$$

And also by the same Definition, $B \wr A$ is the split extension $K \rtimes_\theta A$, so as we said in Lemma 3.1.2, a presentation of $K \rtimes_\theta A$ is given by

$$\mathcal{P}' = \Big\langle \mathbf{y}^{(a)} \;\; (a \in A), \mathbf{x} \; ; \; \mathbf{s}^{(a)} \;\; (a \in A), \mathbf{r}, \, [y^{(a)}, z^{(a')}] \;\; (a, a' \in A, \, a < a', \, y, z \in \mathbf{y}), \, \mathbf{t} \Big\rangle .$$

Here $\mathbf{t} = \{ y^{(a)} x {y^{(a a_x)}}^{-1} x^{-1} \,|\, y \in \mathbf{y}, x \in \mathbf{x} \}$, where for any $c \in A$, $y^{(c)}$ represents the element of $\underbrace{B \times B \times \cdots \times B}_{|A| \, times}$ which has 1 in all positions except position c and the value in position c is b_y where $b_y \in B$. Then $\mathcal{P}'$ is the same presentation as $\mathcal{P}_0$.

Therefore $\mathcal{P}_0$ actually is a presentation of $B \wr A$, as required. □

We will identify G with the group defined by $\mathcal{P}_0$.

Lemma 3.2.4 *If W is a word on* $\mathbf{x}$*, say* $W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$*, then*

$$W^{-1} y^{(a)} W = y^{(a a_{x_1}^{\varepsilon_1} a_{x_2}^{\varepsilon_2} \cdots a_{x_n}^{\varepsilon_n})}$$

in G.

Proof. We will use induction on $L(W)$.

i) Let $L(W) = 1$. Then for $x_1 \in \mathbf{x}$, $y \in \mathbf{y}$ and $a \in A$ we have

$$x_1^{-1} y^{(a a_{x_1}^{-1})} x_1 = y^{((a a_{x_1}) a_{x_1}^{-1})}$$

in G. So

$$x_1 y^{(a)} x_1^{-1} = y^{(a a_{x_1}^{-1})}$$

in G. Hence, for $\varepsilon_1 = \pm 1$ we have

$$x_1^{-\varepsilon_1} y^{(a)} x_1^{\varepsilon_1} = y^{(a a_{x_1}^{\varepsilon_1})}$$

in G.

ii) Assume that the result holds for $L(W) = n - 1$. Now suppose that $L(W) = n$. Then let $W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ ($x_i \in \mathbf{x}$, $\varepsilon_i = \pm 1$ for $1 \leqslant i \leqslant n$). By induction hypothesis, we know that

$$(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{n-1}^{\varepsilon_{n-1}})^{-1} y^{(a)} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{n-1}^{\varepsilon_{n-1}} = y^{(a a_{x_1}^{\varepsilon_1} a_{x_2}^{\varepsilon_2} \cdots a_{x_{n-1}}^{\varepsilon_{n-1}})}$$

in G. Now let us conjugate it by $x_n^{\varepsilon_n}$. Then we get

$$x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \cdots x_1^{-\varepsilon_1} y^{(a)} x_1^{\varepsilon_1} \cdots x_{n-1}^{\varepsilon_{n-1}} x_n^{\varepsilon_n} = x_n^{-\varepsilon_n} y^{(a a_{x_1}^{\varepsilon_1} a_{x_2}^{\varepsilon_2} \cdots a_{x_{n-1}}^{\varepsilon_{n-1}})} x_n^{\varepsilon_n},$$

and by the same process as in the first step, we have

$$x_n^{-\varepsilon_n} y^{(a a_{x_1}^{\varepsilon_1} a_{x_2}^{\varepsilon_2} \cdots a_{x_{n-1}}^{\varepsilon_{n-1}})} x_n^{\varepsilon_n} = y^{(a a_{x_1}^{\varepsilon_1} \cdots a_{x_{n-1}}^{\varepsilon_{n-1}} a_{x_n}^{\varepsilon_n})}.$$

Therefore we have

$$W^{-1} y^{(a)} W = y^{(a a_{x_1}^{\varepsilon_1} \cdots a_{x_{n-1}}^{\varepsilon_{n-1}} a_{x_n}^{\varepsilon_n})}$$

in G, as required. □

- For each $a \in A$, choose a word W_a on $\mathbf{x}$ representing a. (That is, if $W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ then the product $a_{x_1}^{\varepsilon_1} a_{x_2}^{\varepsilon_2} \cdots a_{x_n}^{\varepsilon_n}$ in A is equal to a.) When $a = 1$, choose W_a to be the empty word.

We now perform a sequence of Tietze transformations on $\mathcal{P}_0$.

STEP 1 : Add the relators $y^{(a)} = {W_a}^{-1} \, y^{(1)} \, W_a$ $(a \in A, a \neq 1, y \in \mathbf{y})$ to $\mathcal{P}_0$ since these are consequences of the relators of $\mathcal{P}_0$ by Lemma 3.2.4. Then we obtain a new presentation

$$\begin{aligned}\mathcal{P}_1 \;=\; \Big\langle & \mathbf{y}^{(a)} \;\, (a \in A), \, \mathbf{x} \, ; \; \mathbf{s}^{(a)} \;\, (a \in A), \, \mathbf{r}, \, y^{(a)} z^{(a')} = z^{(a')} y^{(a)} \;\, (a, a' \in A, \, a < a', \, y, z \in \mathbf{y}), \\ & x^{-1} y^{(a)} x = y^{(a a_x)} \;\, (a \in A, \, x \in \mathbf{x}, \, y \in \mathbf{y}), y^{(a)} = {W_a}^{-1} y^{(1)} W_a \;\, (a \in A, \, a \neq 1, \, y \in \mathbf{y}) \Big\rangle \, .\end{aligned}$$

STEP 2 : Delete the relators $\mathbf{s}^{(a)}$ where $a \neq 1$ since these are consequence of the relators $\mathbf{s}^{(1)}$, $x^{-1} y^{(a)} x = y^{(a a_x)}$ $(a \in A, \, x \in \mathbf{x}, \, y \in \mathbf{y})$ and $y^{(a)} = {W_a}^{-1} \, y^{(1)} \, W_a$ $(a \in A, \, a \neq 1, \, y \in \mathbf{y})$. So after deletion we have just the relators $\mathbf{s}^{(1)}$ in the new presentation. We can show it as follows.

We have the relators

$$y^{(a)} = {W_a}^{-1} \, y^{(1)} \, W_a \qquad (a \neq 1),$$

in $\mathcal{P}_1$. Now let $S^{(1)} \in \mathbf{s}^{(1)}$, so the letters in $S^{(1)}$ belong to $\mathbf{y}^{(1)}$ and similarly let for $a \neq 1$, $S^{(a)} \in \mathbf{s}^{(a)}$, so the letters in $S^{(a)}$ belong to $\mathbf{y}^{(a)}$. And by a conclusion of Lemma 3.2.4, we get

$$S^{(a)} = {W_a}^{-1} \, S^{(1)} \, W_a.$$

Here, since $\mathbf{s}^{(1)}$ is a relator in $\mathcal{P}_1$ then $S^{(1)} = 1$ in G and then the above equation implies that $S^{(a)} = 1$ in G. Therefore we can delete $\mathbf{s}^{(a)}$ where $a \neq 1$ and then we have the presentation

$$\begin{aligned}\mathcal{P}_2 \;=\; \Big\langle & \mathbf{y}^{(a)} \;\, (a \in A), \, \mathbf{x} \, ; \; \mathbf{s}^{(1)}, \, \mathbf{r}, \, y^{(a)} z^{(a')} = z^{(a')} y^{(a)} \;\, (a, a' \in A, \, a < a', \, y, z \in \mathbf{y}), \\ & x^{-1} y^{(a)} x = y^{(a a_x)} \;\, (a \in A, \, x \in \mathbf{x}, \, y \in \mathbf{y}), y^{(a)} = {W_a}^{-1} y^{(1)} W_a \;\, (a \in A, \, a \neq 1, \, y \in \mathbf{y}) \Big\rangle \, .\end{aligned}$$

STEP 3 : Delete the relations $x^{-1}\,y^{(a)}\,x = y^{(aa_x)}$ $(a \in A,\, x \in \mathbf{x},\, y \in \mathbf{y})$ from $\mathcal{P}_2$. We must show that these are the consequence of the other relators of $\mathcal{P}_2$. It can be shown as follows.

Take ${W_a}^{-1}\,y^{(1)}\,W_a = y^{(a)}$ and conjugate it by $x \in \mathbf{x}$. Then we get,

$$x^{-1}\,{W_a}^{-1}\,y^{(1)}\,W_a\,x \;=\; x^{-1}\,y^{(a)}\,x.$$

But $W_a x$ represents aa_x in A, so $W_a x = W_{aa_x}$ in A. (That is, $W_a x$ and W_{aa_x} are equal modulo the relators $\mathbf{r}$.)

Hence, modulo the relators $\mathbf{r}$ we can replace the above by

$${W_{aa_x}}^{-1}\,y^{(1)}\,W_{aa_x} \;=\; x^{-1}\,y^{(a)}\,x,$$

and we thus obtain

$$y^{(aa_x)} \;=\; x^{-1}\,y^{(a)}\,x.$$

Therefore we have the presentation

$$\begin{aligned}\mathcal{P}_3 \;=\; &\Big\langle \mathbf{y}^{(a)}\;\; (a \in A),\, \mathbf{x}\,;\; \mathbf{s}^{(1)},\, \mathbf{r},\, y^{(a)}z^{(a')} = z^{(a')}y^{(a)}\;\; (a, a' \in A,\, a < a',\, y, z \in \mathbf{y})\,, \\ &y^{(a)} = {W_a}^{-1}y^{(1)}W_a\;\; (a \in A,\, a \neq 1,\, y \in \mathbf{y})\Big\rangle\,.\end{aligned}$$

STEP 4 : Delete the generators $\mathbf{y}^{(a)}$ where $a \neq 1$ and replace all $y^{(a)}$, $z^{(a')}$ by ${W_a}^{-1}\,y^{(1)}\,W_a$ and ${W_{a'}}^{-1}\,z^{(1)}\,W_{a'}$ $(a, a' \in A$ and $a, a' \neq 1\, y, z \in \mathbf{y}^{(a)})$ in

$$y^{(a)}\,z^{(a')} \;=\; z^{(a')}\,y^{(a)}.$$

After deletion and replacement we have just the generators $\mathbf{y}^{(1)}$. Then we have the presentation

$$\mathcal{P}_4 = \big\langle \mathbf{y}^{(1)},\, \mathbf{x}\,;\; \mathbf{s}^{(1)},\, \mathbf{r},\, [W_a^{-1}\,y^{(1)}\,W_a\,,\; {W_{a'}}^{-1}\,z^{(1)}\,W_{a'}]\;\; (a, a' \in A,\, a < a',\, y,\, z \in \mathbf{y})\big\rangle\,.$$

STEP 5 : Delete the relators of the form $[W_a^{-1}\,y^{(1)}\,W_a\,,\; {W_{a'}}^{-1}\,z^{(1)}\,W_{a'}]$ $(a, a' \in A,\, 1 < a < a',\, y\,,\, z \in \mathbf{y})$ since these are consequence of the relators of the form $[y^{(1)}\,,\; {W_{a'}}^{-1}\,z^{(1)}\,W_{a'}]$ $(a' \in A,\, a' \neq 1,\, y, z \in \mathbf{y})$ and $\mathbf{r}$. We can show it as follows.

For any $a, a' \in A$ where $1 < a < a'$, take a relator

$$[W_a{}^{-1}\, y^{(1)}\, W_a\,,\; W_{a'}{}^{-1}\, z^{(1)}\, W_{a'}].$$

Then conjugate it by W_a, we get

$$[y^{(1)}\,,\; W_a W_{a'}^{-1} z^{(1)} W_{a'} W_a^{-1}].$$

This is equal to some relator which is of the form

$$[y^{(1)}\,,\; W_{a''}{}^{-1}\, z^{(1)}\, W_{a''}],$$

in presentation $\mathcal{P}_4$, since $W_{a'} W_a^{-1} = W_{a''}$ in A, where $a'' \neq 1$. Thus we have the presentation

$$\mathcal{P}_5 = \left\langle \mathbf{y}^{(1)}, \mathbf{x}\,;\; \mathbf{s}^{(1)}, \mathbf{r}, [y^{(1)}\,,\; W_a{}^{-1}\, z^{(1)}\, W_a]\;\; (a \in A,\, a \neq 1,\, y\,,\, z \in \mathbf{y}) \right\rangle.$$

Note that, from now on, we will omit the superscripts (1) on relators in our presentation, so that $\mathcal{P}_5$ becomes

$$\mathcal{P}_5 = \left\langle \mathbf{y}, \mathbf{x}\,;\; \mathbf{s}, \mathbf{r}, [y\,,\; W_a{}^{-1}\, z\, W_a]\;\; (a \in A,\, a \neq 1,\, y, z \in \mathbf{y}) \right\rangle.$$

Now we will apply some reductions on the $[y\,,\; W_a{}^{-1}\, z\, W_a]$ $(a \in A,\, a \neq 1,\, y\,,\, z \in \mathbf{y})$ relators from $\mathcal{P}_5$. Note that the number of these relators is $(|A| - 1)|\mathbf{y}|^2$.

STEP 6 : The set $A \backslash \{1\}$ can be divided into singletons $\{a\}$ $(a \in A,\, a$ an involution) and pairs $\{a,\, a^{-1}\}$ (a not an involution). Let A^+ be a choice of one element from each pair $\{a,\, a^{-1}\}$. (Note that $|A^+| = \frac{1}{2}(|A| - 1 - m)$.) Let Inv be the set of the involutions in the group A. Now let us delete the commutator relators which involve elements in the set $A \backslash (\{1\} \cup A^+ \cup Inv\})$, since these are consequences of the relators which involve elements in the set $A^+ \cup Inv$. It can be done as follows.

Let $a \in A \backslash (\{1\} \cup A^+ \cup Inv\})$. Let us take a relator $[y\,,\, W_a^{-1}\, z\, W_a]$ $(y, z \in \mathbf{y})$, and let us conjugate it by W_a. (Recall that W_a is a word on $\mathbf{x}$ representing a.) Then we get

$$[W_a\, y\, W_a^{-1}\,,\; z].$$

The inverse of it is

$$[z\,,\, W_a\, y\, W_a^{-1}],$$

which can be written as

$$[z\,,\, (W_a^{-1})^{-1}\, y\, W_a^{-1}].$$

Thus, since $W_a^{-1} = W_{a^{-1}}$ in A, then we get

$$[z\,,\, (W_{a^{-1}})^{-1}\, y\, W_{a^{-1}}],$$

where $a^{-1} \in A^+$.

After deletion, we have the presentation

$$\mathcal{P}_6 = \left\langle \mathbf{y},\, \mathbf{x}\;;\; \mathbf{s},\, \mathbf{r},\, [y\,,\, W_a^{-1}\, z\, W_a]\ \ (a \in A^+ \cup Inv,\ y, z \in \mathbf{y}) \right\rangle.$$

Now, we can still apply some reductions on the relators $[y\,,\, W_a^{-1}\, z\, W_a]$ $(a \in A^+ \cup Inv,\ y, z \in \mathbf{y})$. Note that, the number of these relators is

$$\frac{1}{2}(|A| - 1 + m)|\mathbf{y}|^2.$$

Let us choose an ordering $y_1 < y_2 < \cdots < y_n$ of the elements of the generating set $\mathbf{y}$.

STEP 7 : Delete the relators of the form $[z\,,\, W_a^{-1}\, y\, W_a]$ $(a \in Inv,\ y, z \in \mathbf{y},\ y < z)$ since these are consequences of the relators of the form $[y\,,\, W_a^{-1}\, z\, W_a]$ $(a \in Inv,\ y, z \in \mathbf{y},\ y < z)$. It can be shown as follows.

Let $a \in Inv$ and $y, z \in \mathbf{y}$, where $y < z$. Let us take a relator $[y\,,\, W_a^{-1}\, z\, W_a]$, and let us conjugate it by W_a. Then we get

$$[W_a\, y\, W_a^{-1},\, z].$$

The inverse of it is,

$$[z\,,\, W_a\, y\, W_a^{-1}].$$

But, since $a \in Inv$ then we have $W_a = W_a^{-1}$ in A. So, we get

$$[z\,,\, W_a^{-1}\, y\, W_a],$$

as required.

Then we have the presentation

$$\begin{aligned}\mathcal{P}_7 &= \big\langle \mathbf{y},\, \mathbf{x}\,;\, \mathbf{s},\, \mathbf{r},\, [y\,,\, W_a^{-1}\, z\, W_a]\ \ (a \in A^+,\, y, z \in \mathbf{y}),\\ &\qquad [y\,,\, W_a^{-1}\, z\, W_a]\ \ (a \in Inv,\, y, z \in \mathbf{y},\, y \leqslant z)\big\rangle\,.\end{aligned}$$

Now the number of relators $[y\,,\, W_a^{-1}\, z\, W_a]$ $(a \in A^+,\, y, z \in \mathbf{y})$ is $\frac{1}{2}(|A|-1-m)|y^2|$ and the number of relators $[y\,,\, W_a^{-1}\, z\, W_a]$ $(a \in Inv,\, y, z \in \mathbf{y},\, y \leqslant z)$ is $m|\mathbf{y}|^2 - \frac{|\mathbf{y}|(|\mathbf{y}|-1)}{2}m$. So, we have

$$\frac{1}{2}|\mathbf{y}|^2(|A| - 1 + \frac{m}{|\mathbf{y}|})$$

commutator relators in $\mathcal{P}_7$.

Therefore the Euler characteristic of the presentation $\mathcal{P}_7$ can be computed as follows.

$$\begin{aligned}\chi(\mathcal{P}_7) &= 1 - (|\mathbf{x}| + |\mathbf{y}|) + |\mathbf{r}| + |\mathbf{s}| + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|})\\ &= 1 - (|\mathbf{x}| + |\mathbf{y}|) + 1 - 1 + |\mathbf{r}| + |\mathbf{s}| + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|})\\ &= (1 - |\mathbf{x}| + |\mathbf{r}|) + (1 - |\mathbf{y}| + |\mathbf{s}|) - 1 + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|})\\ &= \chi(\mathcal{P}_A) + \chi(\mathcal{P}_B) - 1 + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|})\\ &= \delta(A) + \delta(B) - 1 + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|})\\ &\quad (\text{ since } \mathcal{P}_A \text{ and } \mathcal{P}_B \text{ are efficient presentation })\\ &= 1 + d(H_2(A)) + 1 + d(H_2(B)) - 1 + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|})\\ &= d(H_2(A)) + d(H_2(B)) + 1 + \frac{1}{2}\,|\mathbf{y}|^2\,(|A| - 1 + \frac{m}{|\mathbf{y}|}).\end{aligned}$$

Note that, if $Inv = \emptyset$ then $m = 0$, so that the Euler characteristic of $\mathcal{P}_7$ becomes

$$\chi(\mathcal{P}_7) = d(H_2(A)) + d(H_2(B)) + 1 + \frac{1}{2}\,(|A| - 1)\,|\mathbf{y}|^2.$$

And then, by the assumption $|\mathbf{y}| = d(B) = n$ and by equations (3.4), (3.5), we have

$$\chi(\mathcal{P}_7) = \delta(G).$$

Therefore $\mathcal{P}_7$ is an efficient presentation for the group $G = B \wr A$.

Lemma 3.2.5 *Suppose that* $g = d(A) = d(H_1(A))$. *If* $(t(H_1(A)), t(H_1(B))) \neq 1$ *then*

$$d(G) = g + n.$$

Proof. Now, let us take the presentation $\mathcal{P}_7$ for the group G. Since $\mathcal{P}_7$ has $g+n$ generators then we certainly have

$$d(G) \leqslant g + n.$$

So we just need to show that $g + n \leqslant d(G)$. To do that, we will use the fact that the minimal number of generators of a group is greater than or equal to the minimal number of generators of a quotient of that group, in particular, $d(G) \geqslant d(G^{ab})$. So, we will show that $d(G^{ab}) = g + n$.

Now let us choose an ordering $x_1 < x_2 < \cdots < x_g$ of the elements of the generating set $\mathbf{x}$.

The first homology group of G can be given as follows.

$$\begin{aligned} G^{ab} \;=\; & \big\langle \mathbf{y}, \mathbf{x}\,;\, \mathbf{s}, \mathbf{r}, [y\,,\, W_a^{-1}\, z\, W_a] \;\; (a \in A^+, \, y, z \in \mathbf{y}), \\ & [y\,,\, W_a^{-1}\, z\, W_a] \;\; (a \in Inv, \, y, z \in \mathbf{y}, \, y \leqslant z), \, [y, \, x] \;\; (y \in \mathbf{y}, \, x \in \mathbf{x}), \\ & [y, \, z] \;\; (y \in \mathbf{y}, \, y < z), \, [x, \, x'] \;\; (x, \, x' \in \mathbf{x}, \, x < x')\big\rangle\,. \end{aligned}$$

By applying deletion operations to this presentation of G^{ab}, we have that

$$\begin{aligned} G^{ab} \;&=\; \langle \mathbf{y}, \mathbf{x}\,;\, \mathbf{s}, \mathbf{r}, [y, \, z] \;\; (y, \, z \in \mathbf{y}, \, y < z), \, [x, \, x'] \;\; (x, \, x' \in \mathbf{x}, \, x < x'), \\ & \qquad [y, \, x] \;\; (y \in \mathbf{y}, \, x \in \mathbf{x})\rangle \\ &\cong\; A^{ab} \oplus B^{ab} \\ &=\; H_1(A) \oplus H_1(B). \end{aligned}$$

And so, by Proposition 3.1.10 and by the assumption $(t(H_1(A)), t(H_1(B))) \neq 1$, we have that

$$d(G^{ab}) = d(H_1(A)) + d(H_1(B)).$$

Also, by the assumptions $d(H_1(A)) = d(A) = g$ and $d(H_1(B)) = d(B) = n$, we get that

$$d(G^{ab}) = g + n,$$

as required. □

3.3 Examples and applications

Example 3.3.1 *Let us take the metacyclic group*

$$B = \left\langle a, b\,;\, a^{10}, b^2, bab^{-1} = a^{-1} \right\rangle$$

which has order 20*. Then, by* [38]*,* $H_2(B) = \mathbb{Z}_2$*. So, we can see by a simple calculation, the above presentation of* B *is efficient. After that, if we find the abelianization group* $H_1(B)$ *of* B *and then if we apply some Tietze transformations on the presentation of* $H_1(B)$*, we get*

$$\begin{aligned} H_1(B) &= \left\langle a, b\,;\, a^2, b^2, [a,b] \right\rangle \\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2. \end{aligned}$$

So, $d(H_1(B)) = d(B)$*. Then,*

$$t(H_1(B)) = 2 = t(H_2(B)).$$

Hence by Theorem 3.2.1*, if* A *is a finite group such that* $|A|$ *is even and* $2 \mid t(H_2(A))$*, and if* A *has an efficient presentation then* $B \wr A$ *has an efficient presentation. Moreover, if* A *has an efficient presentation on* $g = d(A) = d(H_1(A))$ *generators then* $B \wr A$ *has an efficient presentation on* $d(B \wr A) = 2 + g$ *generators.* ◊

Example 3.3.2 *Now, let*

$$B = \left\langle a, b\,;\, a^3, b^3, (ab)^3, (a^{-1}b)^3 \right\rangle.$$

By [38]*,* B *has order* 27*. And again by* [38]*,*

$$H_2(B) = \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Then,

$$\delta(B) = 1 + d(H_2(B)) = 3.$$

On the other hand, the Euler characteristic of the above presentation is $1-2+4=3$. *Therefore* B *has an efficient presentation on* 2 *generators. Also, the first homology group of* B *is*

$$H_1(B) = \left\langle a, b\,;\, a^3, b^3, (ab)^3, (a^{-1}b)^3, [a,b] \right\rangle .$$

But, by applying some deletion operations to this presentation of $H_1(B)$*, we have that*

$$\begin{aligned} H_1(B) &= \left\langle a, b\,;\, a^3, b^3, [a,b] \right\rangle \\ &\cong \mathbb{Z}_3 \times \mathbb{Z}_3. \end{aligned}$$

So, $d(H_1(B)) = d(B)$. *Therefore*

$$t(H_1(B)) = 3 = t(H_2(B)).$$

Then, again by Theorem 3.2.1 *and Lemma* 3.2.5*, if* A *is a finite group such that* $|A|$ *is odd and* $3 \mid t(H_2(A))$*, and if* A *has an efficient presentation then* $B \wr A$ *has an efficient presentation. Moreover, if* A *has an efficient presentation on* $g = d(A) = d(H_1(A))$ *generators then* $B \wr A$ *has an efficient presentation on* $d(B \wr A) = 2 + g$ *generators.* $\Diamond$

Lemma 3.3.3 *If* G *is a finite* p*-group, then*

$$\Phi(G) = G'G^p,$$

where $\Phi(G)$ *denotes the Frattini subgroup.*

Proposition 3.3.4 (Burnside Basis Theorem) *Let* X *be a subset of a finite* p*-group* G*. Then* X *generates* G *if and only if the cosets* $\{\, x\Phi(G) : x \in X \,\}$ *generate* $G/\Phi(G)$*. Every minimal set of generators for* G *has the same number of elements.*

Proofs of Lemma 3.3.3 and Proposition 3.3.4 can be found in [35].

Now we can prove the following Proposition, by using these two above well-known results.

Proposition 3.3.5 *Let* B *be an arbitrary finite* p*-group. Then*

$$d(B) = d(H_1(B)).$$

Proof. Let $d(B) = n$, where $n \in \mathbb{Z}^+$. Since $H_1(B) = B/B'$ then we just need to show that $d(B/B') = n$.

By Lemma 3.3.3, we have that

$$\Phi(B) = B'B^p \supseteq B'.$$

So there is a well-defined epimorphism

$$B/B' \longrightarrow B/\Phi(B),$$

and so, $d(B/B') \geqslant d(B/\Phi(B))$. Then by the Burnside Basis Theorem, $d(B/\Phi(B)) = d(B)$. In other words, $d(B/B') \geqslant d(B)$. On the other hand, by the fact that the minimal number of generators of a group is greater than or equal to the minimal number of generators of a quotient of that group, then we have that $d(B) \geqslant d(B/B')$. Therefore,

$$d(B) = d(B/B'),$$

as requried. □

Corollary 3.3.6 *Let A, B be finite p-groups. Suppose B has an efficient presentation on $d(B)$ generators and A has an efficient presentation. Then $B \wr A$ has an efficient presentation. Moreover, if A has an efficient presentation on $d(A)$ generators, then $B \wr A$ has an efficient presentation on $d(B \wr A)$ generators.*

Proof. It is given that they have efficient presentations. And since they are finite p-groups then by Proposition 3.3.5, $d(B) = d(H_1(B))$, and their homology groups are p-groups, as well. So p divides $t(H_2(A))$, $t(H_2(B))$ and $t(H_1(B))$. Therefore by Theorem 3.2.1, $B \wr A$ has an efficient presentation, and then by Lemma 3.2.5, $d(B \wr A) = d(B) + d(A)$, as required. □

Corollary 3.3.7 *Let B be a finite p-group (p odd) and suppose that B has an efficient presentation on $d(B)$ generators.*

If $|A|$ is odd and $p \mid t(H_2(A))$ then $B \wr A$ has an efficient presentation.

It can be proved as Corollary 3.3.6.

Theorem 3.3.8 *Let A be a finite abelian p-group, and let B be a finite p-group which has an efficient presentation on $d(B)$ generators. Then $G = B \wr A$ has an efficient presentation on $d(G)$ generators.*

Again, the proof of this theorem can be obtained by using a similar method to that employed in the proof of Theorem 3.2.1, in conjuction with Lemma 3.2.5.

Corollary 3.3.9 *Let $A_1, A_2, \cdots, A_r, \cdots$ be finite abelian p-groups, and let B be a finite p-group. Let*

$$\begin{aligned} G_0 &= B, \\ G_1 &= G_0 \wr A_1, \\ G_2 &= G_1 \wr A_2, \\ &\vdots \\ G_r &= G_{r-1} \wr A_r. \end{aligned}$$

If B has an efficient presentation on $d(B)$ generators then G_r has an efficient presentation on $d(G_r)$ generators.

Proof. We will use induction on r.

$i)$ Let $r = 1$. Then the result holds by Theorem 3.3.8.

$ii)$ Let $r > 1$ then $G_r = G_{r-1} \wr A_r$. By the induction hypothesis, G_{r-1} has an efficient presentation on $d(G_{r-1})$ generators. Moreover, G_{r-1} is a p-group. Since A_r is an abelian p-group then again, by Theorem 3.3.8, G_r has an efficient presentation on $d(G_r)$ generators. □

Chapter 4

The p-Cockcroft property of the semi-direct products of monoids

4.1 Introduction

In this chapter we introduce the definition of the semi-direct product of any two monoids, a generating set for this product and a presentation of this semi-direct product on the given above generating set, and then we give a trivialiser set (see Chapter 1) of the Squier complex of this presentation, as found by Wang (see [60]). Then we give necessary and sufficient conditions for the standard presentation of the semi-direct product of any two monoids to be p-Cockcroft, for any prime p or 0. Moreover, we give some applications of this to the direct product of two monoids and the semi-direct product of two finite cyclic monoids.

4.2 Monoid presentations

4.2.1 Homomorphisms of monoids defined by presentations

Let $\mathcal{P}$ be a monoid presentation. We will give necessary and sufficient conditions for a function from the generators of $\mathcal{P}$ to a monoid M to induce a homomorphism from the monoid presented by $\mathcal{P}$, say $M(\mathcal{P})$, to the monoid M.

Let M be a monoid, and let $\mathbf{x}$ be a set. Consider a function

$$\psi : \mathbf{x} \longrightarrow M, \quad x \longmapsto m_x. \tag{4.1}$$

For any non-empty positive word W on $\mathbf{x}$, say $W = x_1 x_2 \cdots x_r$, we define

$$\psi(W) = m_{x_1} m_{x_2} \cdots m_{x_r} \quad \text{(product in } M\text{)}.$$

Also, if W is the empty positive word, we define

$$\psi(W) = 1_M.$$

It is clear that ψ is a homomorphism.

Lemma 4.2.1 *Let $\mathcal{P} = [\mathbf{x}\,;\,\mathbf{r}]$ be a monoid presentation. A mapping ψ, as in (4.1), induces a homomorphism*

$$\psi_\star : M(\mathcal{P}) \longrightarrow M, \quad [x]_{\mathcal{P}} \longmapsto m_x$$

if and only if $\psi(R_+) = \psi(R_-)$, for all $R \in \mathbf{r}$.

Proof.

Suppose $\psi(R_+) = \psi(R_-)$ for all $R \in \mathbf{r}$. Let us consider the function

$$\psi_\star : M(\mathcal{P}) \longrightarrow M, \quad [W]_{\mathcal{P}} \longmapsto \psi(W).$$

We must show that this is well-defined. So suppose that $[W_1]_{\mathcal{P}} = [W_2]_{\mathcal{P}}$, where W_1, W_2 are positive words on $\mathbf{x}$.

Special case :

The positive word W_2 is obtained from the positive word W_1 by applying a single elementary operation [see Chapter 1]. So $W_1 = UR_\varepsilon V$, $W_2 = UR_{-\varepsilon}V$ for some positive words U and V on $\mathbf{x}$, $R \in \mathbf{r}$ and $\varepsilon = \pm 1$. Then we have

$$\begin{aligned}
\psi(W_1) &= \psi(UR_\varepsilon V) \\
&= \psi(U)\psi(R_\varepsilon)\psi(V) \\
&= \psi(U)\psi(R_{-\varepsilon})\psi(V) \\
&\qquad \text{since } \psi(R_+) = \psi(R_-) \text{ by assumption} \\
&= \psi(UR_{-\varepsilon}V) \\
&= \psi(W_2).
\end{aligned}$$

General case :

There exists a finite sequence of positive words on $\mathbf{x}$

$$W_1 = U_0,\, U_1,\, \cdots,\, U_n = W_2,$$

where U_{i+1} is obtained from U_i $(0 \leqslant i \leqslant n-1)$ by an elementary operation over monoids. Then by the special case, we have

$$\psi(U_{i+1}) = \psi(U_i).$$

So

$$\psi(W_1) = \psi(U_0) = \cdots = \psi(U_n) = \psi(W_2),$$

as required.

Also $\psi_\star$ is a homomorphism:

$$\begin{aligned} \psi_\star([W_1]_\mathcal{P}\,[W_2]_\mathcal{P}) &= \psi_\star([W_1W_2]_\mathcal{P}) \\ &= \psi(W_1W_2) \\ &= \psi(W_1)\psi(W_2) \\ &\quad \text{since } \psi \text{ is a homomorphism} \\ &= \psi_\star\,[W_1]_\mathcal{P}\,\psi_\star\,[W_2]_\mathcal{P}\,. \end{aligned}$$

Moreover, for all $x \in \mathbf{x}$

$$\psi_\star\,[x] = \psi(x) = m_x.$$

Conversely, suppose that $\psi_\star$ exists. Let $R \in \mathbf{r}$, with $R_+ = x_1x_2\cdots x_n$, $R_- = x'_1x'_2\cdots x'_k$ $(x_i,\, x'_j \in \mathbf{x},\, 1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant k)$. So $[R_+]_\mathcal{P} = [R_-]_\mathcal{P}$, that is, $[x_1x_2\cdots x_n] = [x'_1x'_2\cdots x'_k]$. Then

$$\begin{aligned} & [x_1]\,[x_2]\cdots[x_n] = [x'_1]\,[x'_2]\cdots[x'_k]\,, \\ \Rightarrow\;& \psi_\star([x_1]\,[x_2]\cdots[x_n]) = \psi_\star([x'_1]\,[x'_2]\cdots[x'_k]) \\ \Rightarrow\;& m_{x_1}m_{x_2}\cdots m_{x_n} = m_{x'_1}m_{x'_2}\cdots m_{x'_k} \\ & \text{since } [x]_\mathcal{P} \overset{\psi_\star}{\longmapsto} m_x \text{ and } \psi_\star \text{ is a homomorphism} \\ \Rightarrow\;& \psi(R_+) = \psi(R_-), \end{aligned}$$

as required. $\square$

Let

$$Mat_n(\mathbb{Z}^+) = \{\mathbf{M} : \mathbf{M} \text{ is a } n \times n\text{-matrix with non-negative integer entries}\}.$$

This is a monoid under matrix multiplication where the identity element is the $n \times n$ identity matrix.

Example 4.2.2 *Let $\mathcal{P} = [x, y \,;\, x^2y^3 = yx]$ be a monoid presentation. Let us choose a map*

$$\psi : \{x, y\} \longrightarrow Mat_2(\mathbb{Z}^+), \quad x \longmapsto m_1, \; y \longmapsto m_2,$$

where

$$m_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \;\; \textit{and} \;\; m_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, since $\psi(x^2y^3) = \psi(yx)$ then, by Lemma 4.2.1, ψ induces a homomorphism

$$\psi_\star : M(\mathcal{P}) \longrightarrow Mat_2(\mathbb{Z}^+), \quad [x]_\mathcal{P} \longmapsto m_1, \; [y]_\mathcal{P} \longmapsto m_2.$$

$\Diamond$

Example 4.2.3 *Let $\mathcal{P}$ be as in Example 4.2.2. Let us choose a map*

$$\hat{\psi} : \{x, y\} \longrightarrow Mat_2(\mathbb{Z}^+), \quad x \longmapsto m_1, \; y \longmapsto m_2,$$

where

$$m_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \;\; \textit{and} \;\; m_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, since $\hat{\psi}(x^2y^3) \neq \psi(yx)$ then $\hat{\psi}$ does not induce a homomorphism

$$\hat{\psi_\star} : M(\mathcal{P}) \longrightarrow Mat_2(\mathbb{Z}^+).$$

$\Diamond$

4.2.2 Presentations of given monoids

Definition 4.2.4 *Let M be a monoid, and let $\mathbf{m} = \{m_x : x \in \mathbf{x}\}$ be a generating set for M. We say that a presentation $\mathcal{P} = [\mathbf{x}\,;\,\mathbf{r}]$ is a presentation of M on the generating set $\mathbf{m}$, if the mapping*

$$\psi : \mathbf{x} \longrightarrow M, \quad x \longmapsto m_x$$

induces an **isomorphism**

$$\psi_\star : M(\mathcal{P}) \longrightarrow M, \quad [x] \longmapsto m_x.$$

Example 4.2.5 (Free abelian monoids) *Let the monoid M be $\mathbb{Z}^{+n}$. Recall that $\mathbb{Z}^{+n}$ consist of all n-vectors $v = (v_1, v_2, \cdots, v_n)$ where $v_1, \cdots, v_n$ are non-negative integers. This is a monoid under vector addition where the identity element is $0 = (0, 0, \cdots, 0)$. Then, $\mathbb{Z}^{+n}$ is generated by the elements $m_i = (0, 0, \cdots, 1, 0, \cdots, 0)$ where the integer in the i^{th} position is 1 and all other entries are 0 $(1 \leqslant i \leqslant n)$. Then*

$$\mathcal{P} = [x_i \ (1 \leqslant i \leqslant n)\,;\, x_ix_j = x_jx_i \ (1 \leqslant i < j \leqslant n)] \tag{4.2}$$

is a presentation of $\mathbb{Z}^{+n}$ on the set $\{m_i : 1 \leqslant i \leqslant n\}$. (The proof of this will be given later in this chapter.) $\Diamond$

We now discuss **finite cyclic monoids**. Some of this material may also be found in [32] (see "Monogenic semigroups").

Let M be a finite cyclic monoid of order $k > 1$, generated by m say. Then

$$1,\ m,\ m^2,\ \cdots,\ m^k$$

all belong to M. Since there are $k + 1$ elements in this list then the elements can not all be distinct. So there exists $0 \leqslant p < q \leqslant k$ such that $m^p = m^q$.

Lemma 4.2.6 *If $m^p = m^q$ in M with $0 \leqslant p < q \leqslant k$ then $q = k$.*

Proof. Firstly, we prove by induction on n that $m^n = m^{\alpha(n)}$ for some $0 \leqslant \alpha(n) \leqslant q-1$.

- Let $0 \leqslant n \leqslant q-1$. Then take $\alpha(n) = n$.

- Now suppose $n \geqslant q$, and assume inductively that $m^t = m^{\alpha(t)}$ for some $0 \leqslant \alpha(t) \leqslant q-1$, for all $t < n$. Let us write $n = \lambda q + \mu$ where $\lambda \geqslant 1$, $0 \leqslant \mu < q$. Then

$$\begin{aligned} m^n &= m^{\lambda q+\mu} = m^{\lambda q} m^{\mu} = (m^q)^{\lambda} m^{\mu} \\ &= (m^p)^{\lambda} m^{\mu} \ \ (\text{since } m^p = m^q) \\ &= m^{\lambda p} m^{\mu} = m^{\lambda p+\mu}. \end{aligned}$$

  By inductive hypothesis, since $\lambda p + \mu < n$ then $m^{\lambda p+\mu} = m^{\alpha(\lambda p+n)}$ for some $0 \leqslant \alpha(\lambda p+\mu) \leqslant q-1$. So, $m^n = m^{\alpha(\lambda p+\mu)}$. Then take $\alpha(n) = \alpha(\lambda p+\mu)$. Hence we get

$$m^n = m^{\alpha(n)} \text{ for some } 0 \leqslant \alpha(n) \leqslant q-1.$$

This implies that $M = \{1, m, m^2, \cdots, m^{q-1}\}$. But since $|M| = k$ this means that k must be equal to q.

Hence the result. □

We deduce from this lemma that

i) we have $m^k = m^l$ for some $0 \leqslant l < k$,

ii) the elements of M are $1, m, m^2, \cdots, m^{k-1}$ and since the order of M is k then these elements must all be distinct,

iii) the positive integer l in *i*) is uniquely determined by M, for if there exists $l' \in \mathbb{Z}^+$ ($l' \neq l$, $0 \leqslant l' < k$) such that $m^{l'} = m^k$ then this gives $m^{l'} = m^l$, which contradicts the above lemma.

Lemma 4.2.7 *A presentation for M on the generating set $\{m\}$ is*

$$\mathcal{P}_{k,l} = \left[x \ ; \ x^k = x^l\right]. \tag{4.3}$$

Proof. Let us consider the mapping $x \overset{\psi}{\longmapsto} m$. Then, by Lemma 4.2.1, we get an induced homomorphism

$$\psi_{\star} : M(\mathcal{P}_{k,l}) \longrightarrow M, \quad [x] \longmapsto m,$$

since $\psi(x^k) = \psi(x^l)$ by i). Note that $\psi_\star$ is onto since $m \in Im\psi_\star$. Clearly $\mathcal{P}_{k,l}$ is a complete rewriting presentation, and the irreducible elements are

$$1,\ x,\ x^2,\ \cdots,\ x^{k-1}.$$

Hence the distinct elements of $M(\mathcal{P}_{k,l})$ are $[1]$, $[x]$, $[x^2]$, $\cdots$, $[x^{k-1}]$, and then $|M(\mathcal{P}_{k,l})| = k$. Now if $\psi_\star$ were not injective then $|Im\psi_\star| < |M(\mathcal{P}_{k,l})| = k$. But this gives a contradiction. So $\psi_\star$ is injective, and is thus an isomorphism. $\Box$

We have now proved that any cyclic monoid of order k is isomorphic to $M(\mathcal{P}_{k,l})$ for some $0 \leqslant l < k$.

Now, for any $0 \leqslant l < k$, $M(\mathcal{P}_{k,l})$ *is* a cyclic monoid of order k, generated by $[x]$. We then deduce from this and the previous paragraph that, up to isomorphism, the cyclic monoids of order k are

$$M(\mathcal{P}_{k,l}) \text{ where } l = 0,\ 1,\ \cdots,\ k-1.$$

Lemma 4.2.8 *If $l \neq l'$ then $M(\mathcal{P}_{k,l}) \ncong M(\mathcal{P}_{k,l'})$.*

Proof. Let us assume that $l < l'$, and consider the cyclic group C of order $k - l$, generated by c. There is a homomorphism γ from $M(\mathcal{P}_{k,l})$ onto C, given by $[x]_{\mathcal{P}_{k,l}} \stackrel{\gamma}{\longmapsto} c$.

Now if there were an isomorphism

$$\omega : M(\mathcal{P}_{k,l'}) \longrightarrow M(\mathcal{P}_{k,l})$$

then the composition $\gamma\omega$, say γ' would give a homomorphism from $M(\mathcal{P}_{k,l'})$ onto C. Hence $\gamma'([x]_{\mathcal{P}_{k,l'}})$ would have to be a generator, say $\hat{c}$ of C. But since $[x]^k_{\mathcal{P}_{k,l'}} = [x]^{l'}_{\mathcal{P}_{k,l'}}$ then we would have

$$\hat{c}^k = \gamma'([x]^k_{\mathcal{P}_{k,l'}}) = \gamma'([x]^{l'}_{\mathcal{P}_{k,l'}}) = \hat{c}^{l'},$$

so $\hat{c}^{(k-l')} = 1$ in C. But since $k - l' < k - l$ this contradicts the fact that the order of $\hat{c}$ must be $k - l$.

Hence the result. $\Box$

Let us denote $M(\mathcal{P}_{k,l})$ by $M_{k,l}$. Summarizing all the above, we have

Theorem 4.2.9 *For a fixed $k > 1$ the monoids $M_{k,l}$ $(0 \leqslant l \leqslant k-1)$ are cyclic of order k, and are pairwise non-isomorphic. Any cyclic monoid of order k is isomorphic to $M_{k,l}$ for some l.*

Let us consider the elements of $M_{k,l}$ more closely. Recall that they are the equivalence classes $[x^i]$ $(0 \leqslant i < k)$. For $0 \leqslant i < l$, the equivalence class $[x^i]$ just consist of the single element x^i. However for $i \geqslant l$, the equivalence class $[x^i]$ consist of infinitely many elements which are defined by

$$[x^i] = \{x^{i+q(k-l)} : q = 0, 1, 2, \cdots\}.$$

Example 4.2.10 *Let us take the monoid $M_{5,3}$. The equivalence classes are*

$$[x^0] = \{1\},\ [x^1] = \{x\},\ [x^2] = \{x^2\},$$

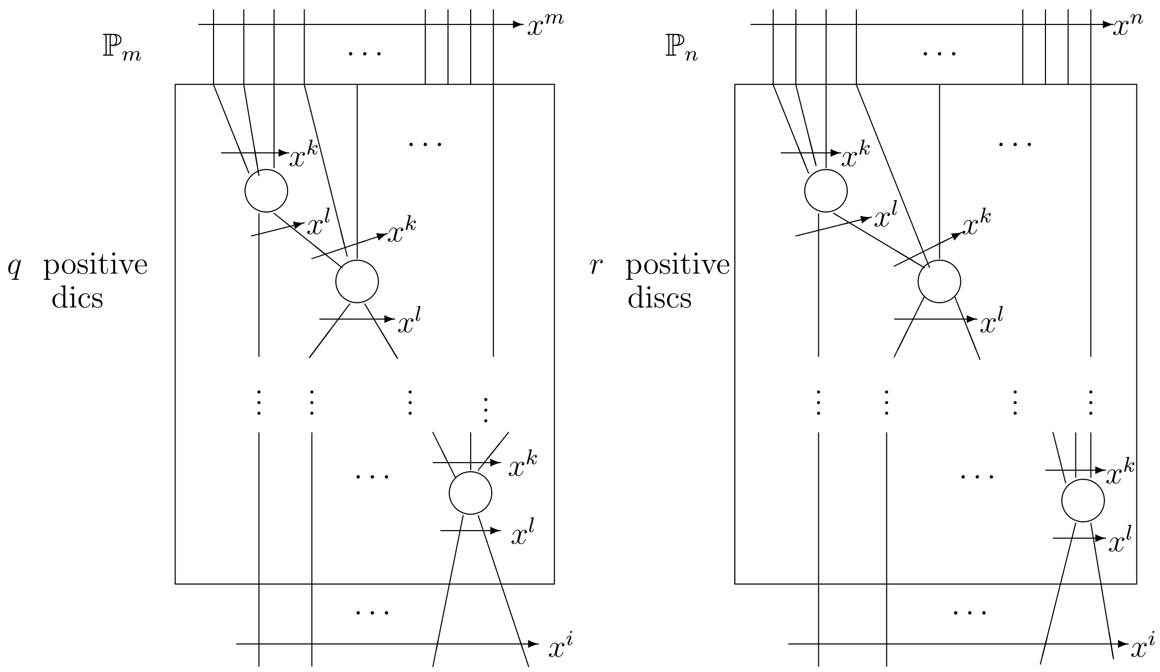
$$[x^3] = \{x^3,\ x^5,\ x^7,\ \cdots\},\ [x^4] = \{x^4,\ x^6,\ x^8,\ \cdots\}.$$

$\diamondsuit$

Suppose m, n $(m \leqslant n)$ belong to the same equivalence class $[x^i]$. If $i < l$ then $m = n$. Suppose $i \geqslant l$. Then we must have

$$m = i + q(k-l) \text{ and } n = i + r(k-l),$$

where q, r are non-negative integers. There will then be a positive path (that is, a monoid picture with all discs labelled by the relator $x^k = x^l$ with sign $+1$) in the Squier complex from x^m to x^i of length q, and similarly from x^n to x^i of length r. This can be illustrated geometrically as follows.



Therefore $\mathbb{P}_n\mathbb{P}_m^{-1}$ is a path from x^n to x^m, and

$$\exp_R(\mathbb{P}_n\mathbb{P}_m^{-1}) = r - q,$$

where R is the relator $x^k = x^l$. Since $r - q = \dfrac{n-m}{k-l}$ then we have

$$\exp_R(\mathbb{P}_n\mathbb{P}_m^{-1}) = \frac{n-m}{k-l}.$$

Note that when $i < l$ (that is, $m = n$) we have the empty path from x^n to x^m.

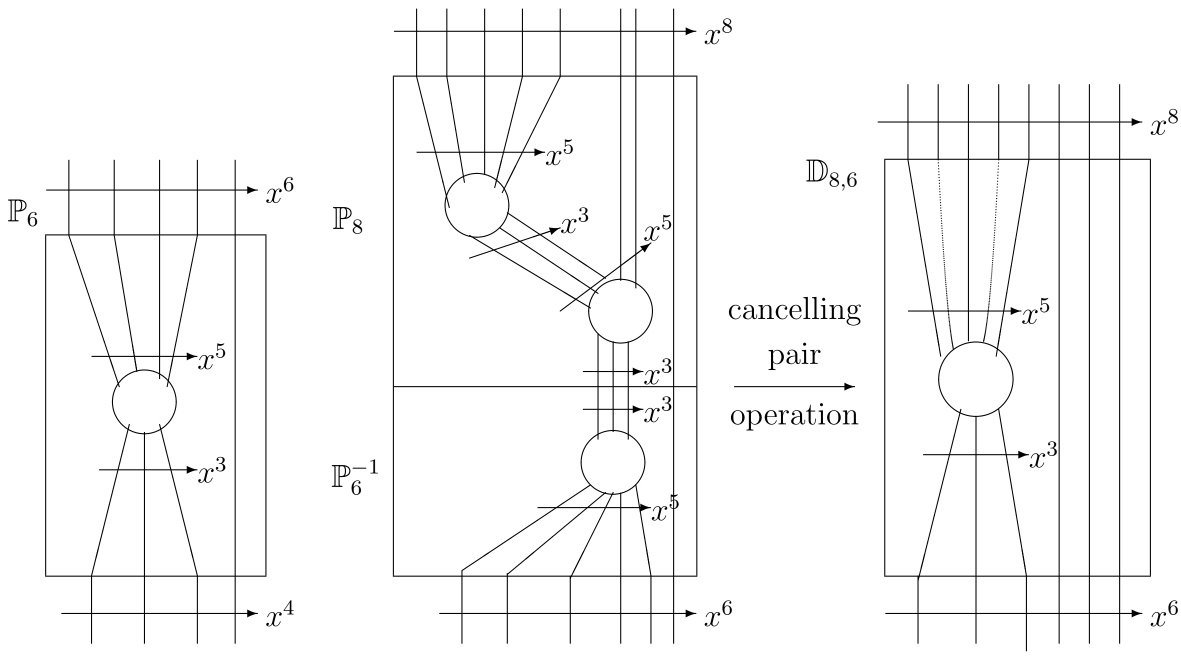
Therefore, we have

Lemma 4.2.11 *Suppose x^m and x^n ($m \leqslant n$) are in the same equivalence class. Then there is a monoid picture $\mathbb{Q}_{n,m}$ with $\iota(\mathbb{Q}_{n,m}) = x^n$, $\tau(\mathbb{Q}_{n,m}) = x^m$ and*

$$\exp_R(\mathbb{Q}_{n,m}) = \frac{n-m}{k-l}.$$

Remark 4.2.12 *Actually, one could take $\mathbb{P}_n$ to be of the form $\mathbb{D}_{n,m}\mathbb{P}_m$, where $\mathbb{D}_{n,m}$ is a path from x^n to x^m, so that $\mathbb{P}_n\mathbb{P}_m^{-1}$ is freely equal to $\mathbb{D}_{n,m}$.*

Example 4.2.10 (continued) *Let us choose $m = 6$ and $n = 8$. Notice that x^6, x^8 are in the same equivalence class $[x^4]$. Then we can show that the picture $\mathbb{P}_8\mathbb{P}_6^{-1}$ is freely equal to $\mathbb{D}_{8,6}$ as in the following figure.*



$\Diamond$

One can give a trivializer set of the Squier complex of $M_{k,l}$ as follows.

Lemma 4.2.13 *Let M be the finite cyclic monoid with the presentation $\mathcal{P}_{k,l}$, as in* (4.3). *Then a trivializer set of the Squier complex $\mathcal{D}(\mathcal{P}_{k,l})$ is given by the pictures $\mathbb{P}^i_{k,l}$* $(1 \leqslant i \leqslant k-1)$*, as in Figure* 4.1.

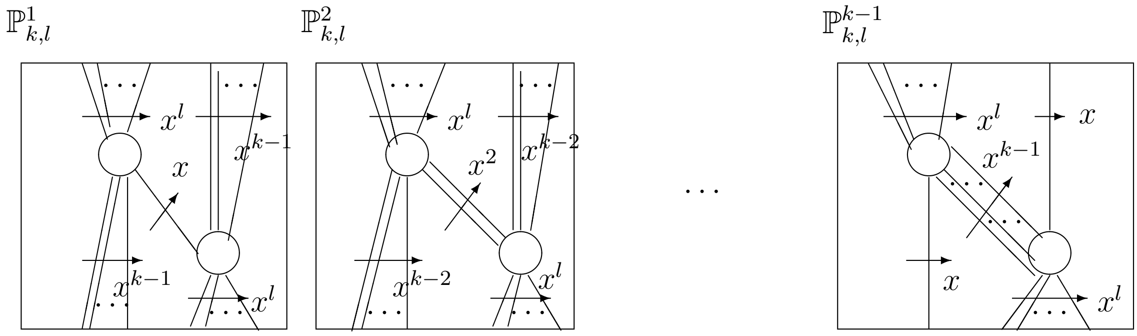


Figure 4.1:

Proof. Since $\mathcal{P}_{k,l}$ is a complete rewriting presentation then, by "overlapping", we can obtain the pictures in Figure 4.1, as required. $\square$

4.2.3 Endomorphisms of monoids

A homomorphism from a monoid to itself is called an *endomorphism*. Let M be a monoid. Then the set of all endomorphisms of M form a monoid under composition, where the identity element is $id : M \longrightarrow M$, and we denote this monoid by $End(M)$.

Let $\mathcal{P} = [\mathbf{x}\ ;\ \mathbf{r}]$ be a presentation of M, that is, $M(\mathcal{P}) \cong M$. For each $x \in \mathbf{x}$, let us consider a map

$$\xi : \mathbf{x} \longrightarrow M(\mathcal{P}), \quad x \longmapsto [W_x]\,,$$

where W_x is a positive word on $\mathbf{x}$. In order to show that this induces a homomorphism, we must use Lemma 4.2.1. For any positive word V on $\mathbf{x}$, say $V = x_1 x_2 \cdots x_n$, let

$$\xi(V) = [W_{x_1} W_{x_2} \cdots W_{x_n}] \quad \text{(product in } M(\mathcal{P})).$$

Then the map ξ induces a homomorphism if and only if

$$\xi(R_+) = \xi(R_-),$$

for all $R \in \mathbf{r}$.

Example 4.2.14 *Let M be $\mathbb{Z}^{+n}$, and let $\mathbf{M}$ be an $n \times n$-matrix with non-negative integral entries. Then we get a mapping*

$$\psi_{\mathbf{M}} : \mathbb{Z}^{+n} \longrightarrow \mathbb{Z}^{+n}, \quad v \longmapsto v\mathbf{M},$$

where $v = (v_1, v_2, \cdots, v_n)$ as in Example 4.2.5. Actually, $\psi_{\mathbf{M}} \in End(\mathbb{Z}^{+n})$ and $\psi_{\mathbf{M}_1}\psi_{\mathbf{M}_2} = \psi_{\mathbf{M}_1\mathbf{M}_2}$. We should note that, if $\phi \in End(\mathbb{Z}^{+n})$ then there exists a matrix $\mathbf{M}$ (depending on ϕ) such that $\phi = \psi_{\mathbf{M}}$.

By the mapping

$$\mathbf{M} \longmapsto \psi_{\mathbf{M}},$$

we get an isomorphism from the monoid $Mat_n(\mathbb{Z}^+)$ to the monoid $End(\mathbb{Z}^{+n})$. $\Diamond$

Example 4.2.15 *Let M be a cyclic monoid generated by m.*

Case 1 : *Suppose M is a cyclic monoid of order k. Then, by Lemma 4.2.7 and Definition 4.2.4, $M_{k,l} \cong M$ where $0 \leqslant l \leqslant k-1$. By Lemma 4.2.1, the mapping*

$$x \overset{\xi}{\longmapsto} [x^i] \quad (0 \leqslant i < k)$$

induces a homomorphism

$$\psi_i : M_{k,l} \longrightarrow M_{k,l}$$

since $[x^{ki}] = [x^{li}]$ in $M_{k,l}$. Moreover, if $\psi : M_{k,l} \longrightarrow M_{k,l}$ is any endomorphism then we must have $\psi([x]) = [x^i]$ for some $0 \leqslant i < k$, so ψ and ψ_i agree on the generating set $\{[x]\}$ of $M_{k,l}$ and so are equal. Hence $\psi_0, \psi_1, \cdots, \psi_{k-1}$ are the only endomorphisms of $M_{k,l}$. Since these endomorphisms take different values at $[x]$ then they are distinct. Hence

$$End(M_{k,l}) = \{\psi_i : i = 0, 1, \cdots, k-1\}.$$

Case 2 : *Suppose M is an infinite cyclic monoid. This means we are working on ${\mathbb{Z}^+}^n$ where $n = 1$, as in Example 4.2.14. So we have*

$$Mat_1(\mathbb{Z}^+) \cong End(M),$$

that is, $\mathbb{Z}^+ \cong End(M)$. $\diamondsuit$

We now consider some **one-relator monoids**.

Example 4.2.16 *Let M be the one-relator monoid with the presentation*

$$\mathcal{P} = \left[x_1,\, x_2 \,;\, x_1x_2^2 = x_2x_1x_2x_1\right].$$

In [21], it has been proved that M has no endomorphism other than the identity homomorphism. $\diamondsuit$

In the next three examples non-trivial endomorphisms will be introduced for some one-relator monoids which will be used later in this thesis.

Example 4.2.16.(a) *Let M be the one-relator monoid given by the presentation $\mathcal{P} = \left[x_1,\, x_2 \,;\, x_1x_2x_1 = x_2x_1^k\right]$. By Lemma 4.2.1, a mapping*

$$\xi : \{x_1, x_2\} \longrightarrow M(\mathcal{P}), \quad x_1 \longmapsto [x_1^i],\ x_2 \longmapsto [x_2],$$

where $i \in \mathbb{Z}^+$*, induces an endomorphism if and only if*

$$[x_1^i x_2 x_1^i] = [x_2 x_1^{ki}].$$

This equality always holds as can be shown as follows.

$$\begin{aligned}
[x_1^i x_2 x_1^i] &= [x_1^{i-1} x_1 x_2 x_1 x_1^{i-1}] = [x_1^{i-1} x_2 x_1^k x_1^{i-1}] \ (\textit{since } x_1 x_2 x_1 = x_2 x_1^k) \\
&= [x_1^{i-2} x_1 x_2 x_1 x_1^{i-2} x_1^k] = [x_1^{i-2} x_2 x_1^k x_1^{i-2} x_1^k] \ (\textit{since } x_1 x_2 x_1 = x_2 x_1^k) \\
&\vdots \quad \vdots \\
&= [x_1^{i-i} x_2 x_1^k x_1^{(i-1)k}] = [x_2 x_1^{ki}].
\end{aligned}$$

$\diamondsuit$

Example 4.2.16.(b) *Let* M *be given by the presentation* $\mathcal{P} = \big[x_1,\, x_2\ ;\ x_1^k x_2 = x_2 x_1^k\big]$*. Again, by Lemma* 4.2.1*, a mapping*

$$\xi : \{x_1, x_2\} \longrightarrow M(\mathcal{P}), \quad x_1 \longmapsto [x_1^i],\ \ x_2 \longmapsto [x_2^j],$$

where $i,\, j \in \mathbb{Z}^+$*, induces an endomorphism if and only if*

$$[x_1^{ki} x_2^j] = [x_2^j x_1^{ki}].$$

Indeed,

$$\begin{aligned}
[x_1^{ki} x_2^j] &= [x_1^{ki-k} x_1^k x_2 x_2^{j-1}] = [x_1^{ki-k} x_2 x_1^k x_2^{j-1}] \ (\textit{since } x_1^k x_2 = x_2 x_1^k) \\
&= [x_1^{ki-2k} x_1^k x_2 x_1^k x_2 x_2^{j-2}] = [x_1^{ki-2k} x_2 x_1^k x_2 x_1^k x_2^{j-2}] \ (\textit{since } x_1^k x_2 = x_2 x_1^k) \\
&\vdots \quad \vdots \\
&= [x_1^{ki-ik} x_2^{j-1} x_1^k x_2 x_1^{(i-1)k}] = [x_2^{(j-1)} x_2 x_1^k x_1^{(i-1)k}] \ (\textit{since } x_1^k x_2 = x_2 x_1^k) \\
&= [x_2^j x_1^{ki}].
\end{aligned}$$

$\diamondsuit$

Example 4.2.16.(c) *Let* M *be given by the presentation* $\mathcal{P} = \big[x_1,\, x_2\ ;\ x_1 x_2 = x_2 x_1^k\big]$*. As previously, by Lemma* 4.2.1*, a mapping*

$$\xi : \{x_1, x_2\} \longrightarrow M(\mathcal{P}), \quad x_1 \longmapsto [x_1^i],\ \ x_2 \longmapsto [x_2],$$

where $i \in \mathbb{Z}^+$, induces an endomorphism if and only if

$$[x_1^i x_2] = [x_2 x_1^{ki}].$$

This always holds as can be shown as follows.

$$\begin{aligned}
[x_1^i x_2] &= [x_1^{i-1} x_1 x_2] = [x_1^{i-1} x_2 x_1^k] \ (\textit{since } x_1 x_2 = x_2 x_1^k) \\
&= [x_1^{i-2} x_1 x_2 x_1^k] = [x_1^{i-2} x_2 x_1^k x_1^k] \ (\textit{since } x_1 x_2 = x_2 x_1^k) \\
&\vdots \quad \vdots \\
&= [x_1^{i-(i-1)} x_2 x_1^k \cdots x_1^k] = [x_1 x_2 x_1^k \cdots x_1^k] = [x_1 x_2 x_1^{k(i-1)}] \\
&= [x_2 x_1^k x_1^{k(i-1)}] \ (\textit{since } x_1 x_2 = x_2 x_1^k) \\
&= [x_2 x_1^{ki}].
\end{aligned}$$

$\Diamond$

4.3 Semi-direct products of monoids

4.3.1 The definition

Let A and K be monoids, and let us take a monoid homomorphism

$$\theta : A \longrightarrow End(K), \quad a \longmapsto \theta_a \ \ (a \in A), \quad 1 \longmapsto id_{End(K)}. \tag{4.4}$$

Then we can define the semi-direct product D of K by A, as follows.

The elements of D are all ordered pairs (a, k) where $a \in A$, $k \in K$ and the product is given by

$$(a, k)(a', k') = (aa', \, (k\theta_{a'})k'). \tag{4.5}$$

By checking the monoid axioms, we can show that D is a monoid as follows.

a) The *closure* holds by (4.5).

b) The *associativity*:

Let a_1, a_2, $a_3 \in A$ and k_1, k_2, $k_3 \in K$. Then we will check whether the equality

$$(a_1, k_1)[(a_2, k_2)(a_3, k_3)] = [(a_1, k_1)(a_2, k_2)](a_3, k_3)$$

holds. Let LHS and RHS be the left hand side and the right hand side of this above equality, respectively. Then we get

$$\begin{aligned} LHS &= (a_1, k_1)(a_2a_3, (k_2\theta_{a_3})k_3) \text{ by (4.5)} \\ &= (a_1(a_2a_3), (k_1\theta_{a_2a_3})(k_2\theta_{a_3})k_3) \text{ by (4.5)} \\ &= (a_1a_2a_3, (k_1\theta_{a_2}\theta_{a_3})(k_2\theta_{a_3})k_3) \text{ since } \theta \text{ is a homomorphism,} \end{aligned}$$

and

$$\begin{aligned} RHS &= (a_1a_2, (k_1\theta_{a_2})k_2)(a_3, k_3) \text{ by (4.5)} \\ &= ((a_1a_2)a_3, (((k_1\theta_{a_2})k_2)\theta_{a_3})k_3) \text{ by (4.5)} \\ &= (a_1a_2a_3, (k_1\theta_{a_2}\theta_{a_3})(k_2\theta_{a_3})k_3) \text{ since } \theta_{a_3} \text{ is a homomorphism.} \end{aligned}$$

So, the associativity holds.

$c)$ The *identity*:

Let 1_A and 1_K be the identity elements of A and K, respectively. Then the identity element of D is $(1_A, 1_K)$. That is, for all $(a, k) \in D$, we need to show that

$$(1_A, 1_K)(a, k) = (a, k) = (a, k)(1_A, 1_K).$$

First of all, we get

$$(1_A, 1_K)(a, k) = (1_Aa, (1_K\theta_a)k),$$

by (4.5). Now, since $\theta_a : K \longrightarrow K$ $(a \in A)$ is a homomorphism then θ_a maps the identity element of K which is 1_K to itself. So, $(1_K\theta_a)k = k$ for all $k \in K$. Thus, $(1_Aa, (1_K\theta_a)k) = (a, k)$.

On the other hand, we get

$$(a, k)(1_A, 1_K) = (a1_A, (k\theta_{1_A})1_K),$$

by (4.5). Since $1_A \in A$ then $\theta_{1_A} \in End(K)$. Furthermore, since $\theta_{1_A} = id_{End(K)}$ then θ_{1_A} is the identity homomorphism of K. Then, for all $k \in K$, $k\theta_{1_A} = k$. Thus, $(a1_A, (k\theta_{1_A})1_K) = (a, k)$.

Therefore D is a monoid.

Remark 4.3.1 *For any $(a, k) \in D$ where $a \in A$ and $k \in K$, we have*

$$(a, k) = (a, 1_K)(1_A, k). \tag{4.6}$$

(To see this let us take $(a, 1_K)(1_A, k)$. Then, by (4.5), we get $(a1_A, (1_K\theta_{1_A})k)$. Since θ_{1_A} is the identity homomorphism and 1_K is the identity element of K then we get (a, k), as required.)

4.3.2 A generating set for D

Let us choose generating sets

$$\mathbf{k} = \{k_y : y \in \mathbf{y}\} \quad \text{and} \quad \mathbf{a} = \{a_x : x \in \mathbf{x}\}$$

for the monoids K and A, respectively. Then the set

$$\mathbf{d} = \{(1, k_y) \ (y \in \mathbf{y}), \ (a_x, 1) \ (x \in \mathbf{x})\}$$

generates D. That is to say, any element in D, say (a, k) where $a \in A$, $k \in K$, can be written as a product of some elements from the set $\mathbf{d}$. We need to show that

$$(a, k) = d_1 d_2 \cdots d_r \text{ where } d_i \in \mathbf{d},\ 1 \leqslant i \leqslant r.$$

Since $\mathbf{a}$ generates A and $\mathbf{k}$ generates K, we have

$$a = a_{x_1} a_{x_2} \cdots a_{x_m} \text{ and } k = k_{y_1} k_{y_2} \cdots k_{y_n}, \tag{4.7}$$

where $x_i \in \mathbf{x}$, $y_j \in \mathbf{y}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and m, $n \geqslant 0$.

Thus,

$$\begin{aligned}(a,k) &= (a,1)(1,k) \text{ by (4.6)},\\ &= (a_{x_1}a_{x_2}\cdots a_{x_m},1)(1,k_{y_1}k_{y_2}\cdots k_{y_n}) \text{ by (4.7)},\\ &= (a_{x_1},1)(a_{x_2},1)\cdots(a_{x_m},1)(1,k_{y_1})(1,k_{y_2})\cdots(1,k_{y_n}) \text{ by (4.6) and}\\ &\quad \text{by the fact that } \theta_{a_{x_i}} \text{ maps the identity element of } K \text{ to itself,}\end{aligned}$$

and then, since each of these pairs is in the set of **d** then we get what we required.

4.3.3 A presentation for D

Let $\mathcal{P}_K = [\mathbf{y}\,;\,\mathbf{s}]$ and $\mathcal{P}_A = [\mathbf{x}\,;\,\mathbf{r}]$ be presentations for K, A on the generating sets **k**, **a**, respectively. Then, by Definition 4.2.4, we have isomorphisms

$$\psi_{K_\star} : M(\mathcal{P}_K) \longrightarrow K, \quad [y]_{\mathcal{P}_K} \longmapsto k_y$$

$$\psi_{A_\star} : M(\mathcal{P}_A) \longrightarrow A, \quad [x]_{\mathcal{P}_A} \longmapsto a_x$$

induced by the functions

$$\psi_K : \mathbf{y} \longrightarrow K, \quad y \longmapsto k_y,$$

$$\psi_A : \mathbf{x} \longrightarrow A, \quad x \longmapsto a_x.$$

For each $y \in \mathbf{y}$, $x \in \mathbf{x}$, let $y\theta_x$ denote a positive word on **y** representing the element $k_y\theta_{a_x}$ of K, that is $\psi_{K_\star}[y\theta_x]_{\mathcal{P}_K} = k_y\theta_{a_x}$. Let T_{yx} denote the relator $yx = x(y\theta_x)$, and let **t** be the set of all relators of the form T_{yx} $(x \in \mathbf{x},\, y \in \mathbf{y})$.

The proof of the following theorem can be found in [55].

Theorem 4.3.2 *A presentation for D on the generating set* **d** *is given by*

$$\mathcal{P}_D = [\mathbf{x},\, \mathbf{y}\;;\; \mathbf{r},\, \mathbf{s},\, \mathbf{t}]\,. \tag{4.8}$$

Remark 4.3.3 *If $W = y_1y_2\cdots y_m$ is a positive word on* **y** *then for any $x \in \mathbf{x}$, we denote the positive word $(y_1\theta_x)(y_2\theta_x)\cdots(y_m\theta_x)$ by $W\theta_x$. If $U = x_1x_2\cdots x_n$ is a positive word on* **x** *then for any $y \in \mathbf{y}$, we denote the positive word $(\cdots((y\theta_{x_1})\theta_{x_2})\theta_{x_3}\cdots)\theta_{x_n})$ by $y\theta_U$, and this can be represented by a picture, say $\mathbb{A}_{U,y}$, as in Figure* 4.2.

$\mathbb{A}_{U,y}$

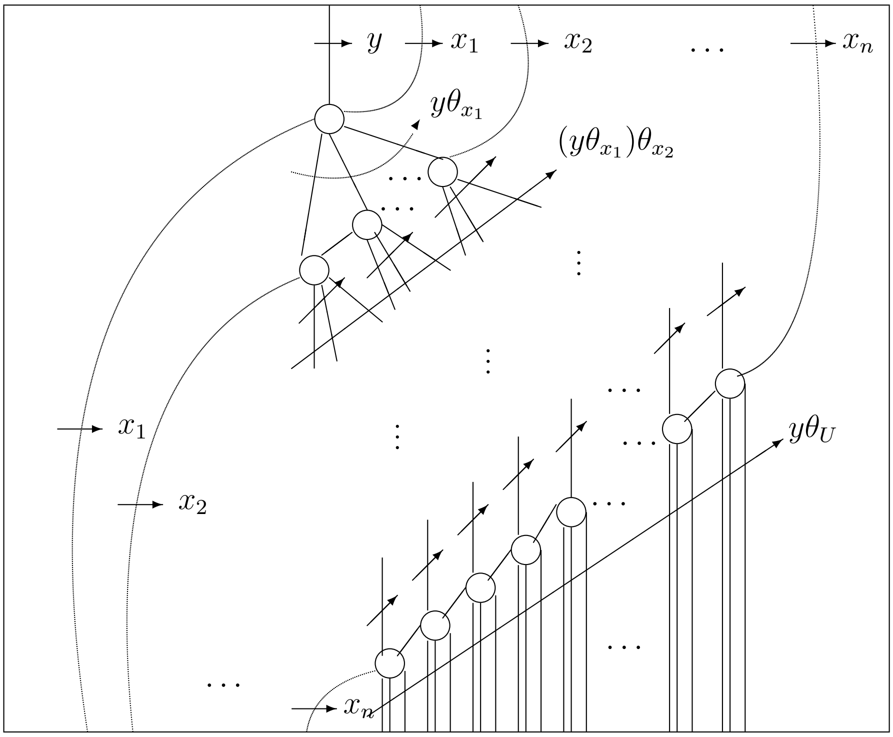


Figure 4.2:

4.3.4 Trivializer of the Squier complex $\mathcal{D}(\mathcal{P}_D)$

Let $\mathbf{X_A}$ and $\mathbf{X_K}$ be trivialiser sets of $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, respectively.

Let $S \in \mathbf{s}$, $x \in \mathbf{x}$. Since $[S_+\theta_x]_{\mathcal{P}_K} = [S_-\theta_x]_{\mathcal{P}_K}$, there is a non-spherical picture, say $\mathbb{B}_{S,x}$, over $\mathcal{P}_K$ with

$$\iota(\mathbb{B}_{S,x}) = S_+\theta_x \ \text{ and } \ \tau(\mathbb{B}_{S,x}) = S_-\theta_x.$$

Note that, there are various $\mathbb{B}_{S,x}$ pictures which can be drawn.

Let $R \in \mathbf{r}$, $y \in \mathbf{y}$. Then we get non-spherical pictures $\mathbb{A}_{R_+,y}$ and $\mathbb{A}_{R_-,y}$, respectively, as in Figure 4.2. We should note that, these pictures consist of only T_{yx} discs ($x \in \mathbf{x}$). Moreover, since $\left[y\theta_{R_+}\right]_{\mathcal{P}_K} = \left[y\theta_{R_-}\right]_{\mathcal{P}_K}$, there is a non-spherical picture, say $\mathbb{C}_{y,\theta_R}$, over $\mathcal{P}_K$ with

$$\iota(\mathbb{C}_{y,\theta_R}) = y\theta_{R_+} \ \text{ and } \ \tau(\mathbb{C}_{y,\theta_R}) = y\theta_{R_-}.$$

We should also note that there are various $\mathbb{C}_{y,\theta_R}$ pictures which can be drawn.

Our aim is now to construct spherical monoid pictures by using these above non-spherical pictures.

Let us take a single $\mathbb{B}_{S,x}$ picture. If we process the initial positive word of $\mathbb{B}_{S,x}$, which is $S_+\theta_x$, and the terminal positive word of $\mathbb{B}_{S,x}$, which is $S_-\theta_x$, by a single x-arc, then we get some T_{yx} $(y \in \mathbf{y})$ discs at the top (and at the bottom) of the $\mathbb{B}_{S,x}$ picture. Then we have a new picture containing a single $\mathbb{B}_{S,x}$ picture and some T_{yx} $(y \in \mathbf{y})$ discs. But for this picture, we get the positive words S_+x (at the bottom) and S_-x (at the top), respectively, that is, it is a non-spherical picture. So, to get a spherical monoid picture from this non-spherical picture, we must fix a single S-disc on the top (or bottom) of this non-spherical picture. Then we have a spherical monoid picture, call it $\mathbb{P}_{S,x}$, as shown in Figure 4.3.

Now let us take the pictures $\mathbb{A}_{R_+,y}$ and $\mathbb{A}^{-1}_{R_-,y}$. We can combine these two pictures by

- fixing a single R-disc between them, and then
- fixing a single $\mathbb{C}_{y,\theta_R}$ picture between the positive words $y\theta_{R_+}$ and $y\theta_{R_-}$, respectively.

Then we get a new picture, and for this picture, we get the positive words yR_+ (at the top) and yR_- (at the bottom), respectively. To get a spherical monoid picture, we must fix a single R-disc on the top (or bottom) of this picture. Then we have a spherical monoid picture, say $\mathbb{P}_{R,y}$, as in Figure 4.3.

Let

$$\mathbf{C_1} = \{\mathbb{P}_{S,x} \,:\, S \in \mathbf{s},\, x \in \mathbf{x}\} \text{ and } \mathbf{C_2} = \{\mathbb{P}_{R,y} \,:\, R \in \mathbf{r},\, y \in \mathbf{y}\}.$$

The proof of the following theorem can be found in [60].

Theorem 4.3.4 *Suppose that $D = K \rtimes_\theta A$ is a semi-direct product with associated presentation $\mathcal{P}_D$, as in* (4.8)*. Let $\mathbf{X_A}$ and $\mathbf{X_K}$ be trivialiser sets of the Squier complexes $\mathcal{D}(\mathcal{P}_A)$ and $\mathcal{D}(\mathcal{P}_K)$, respectively. Then a trivialiser set of $\mathcal{D}(\mathcal{P}_D)$ is*

$$\mathbf{X_A} \cup \mathbf{X_K} \cup \mathbf{C_1} \cup \mathbf{C_2}. \tag{4.9}$$

Let us denote the set (4.9) by $\mathbf{X_D}$.

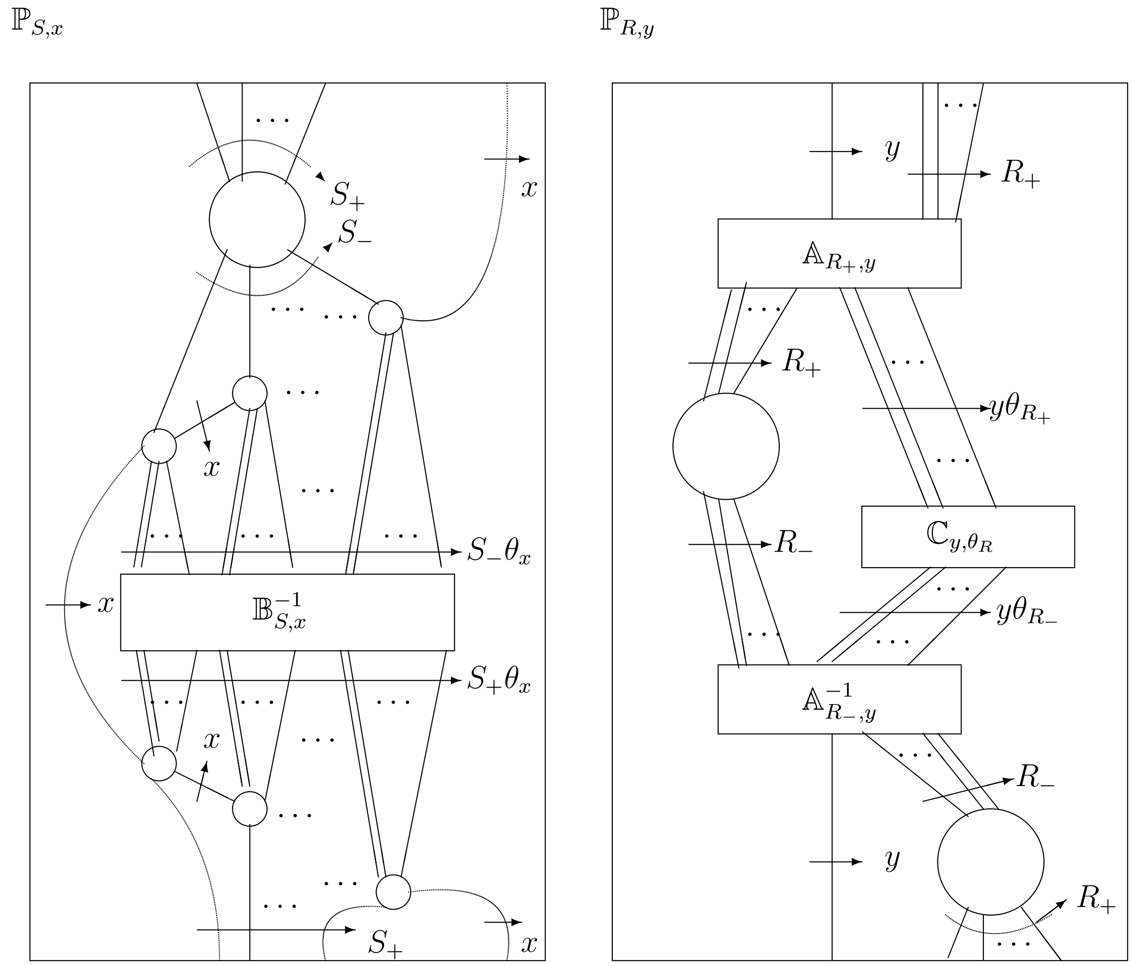


Figure 4.3:

4.3.5 Defining a homomorphism $\theta : A \longrightarrow End(K)$

Suppose that K and A are given by presentations $\mathcal{P}_K = [\mathbf{y}\ ;\ \mathbf{s}]$ and $\mathcal{P}_A = [\mathbf{x}\ ;\ \mathbf{r}]$, respectively. We have seen in Section 4.2.3 how to obtain endomorphisms of K. Let us suppose that, for each $x \in \mathbf{x}$, we have obtained an endomorphism ψ_x of K in this way. So we have a mapping

$$\mathbf{x} \longrightarrow End(K), \quad x \longmapsto \psi_x.$$

In order to show that this induces a homomorphism

$$\theta : A \longrightarrow End(K),$$

we must use the basic Lemma 4.2.1. For any positive word W on $\mathbf{x}$, say $W = x_1 x_2 \cdots x_n$, let

$$\psi_W = \psi_{x_1}\psi_{x_2}\cdots\psi_{x_n} \quad \text{(product in } End(K)).$$

Then the above map induces a homomorphism θ if and only if

$$\psi_{R_+} = \psi_{R_-},$$

for all $R \in \mathbf{r}$. Since two endomorphisms of K agree if and only if they agree on a generating set, we must show that

$$[y]\psi_{R_+} = [y]\psi_{R_-},$$

for all $y \in \mathbf{y}$, $R \in \mathbf{r}$.

Example 4.3.5 *Let K be $\mathbb{Z}^{+n}$. Let us consider the standard presentation* (4.2) *of $\mathbb{Z}^{+n}$, and then let $\mathbf{y}$ be the set of generators and $\mathbf{r}$ be the set of relators of this presentation. Then $\mathcal{P}_K = [\mathbf{y}\ ;\ \mathbf{r}]$ becomes a presentation of the monoid $\mathbb{Z}^{+n}$.*

In Example 4.2.14*, we showed that $Mat_n(\mathbb{Z}^+) \cong End(\mathbb{Z}^+)$. So the endomorphism ψ_x $(x \in \mathbf{x})$ will be $\psi_{\mathbf{M}_x}$ for some matrix $\mathbf{M}_x$. For any positive word $W = x_1 x_2 \cdots x_n$ on $\mathbf{x}$, let $\mathbf{M}_W$ be the product $\mathbf{M}_{x_1}\mathbf{M}_{x_2}\cdots\mathbf{M}_{x_n}$ of the matrices $\mathbf{M}_{x_1}, \cdots, \mathbf{M}_{x_n}$. Then the mapping $x \longmapsto \psi_{\mathbf{M}_x}$ $(x \in \mathbf{x})$ induces a homomorphism*

$$\theta : A \longrightarrow End(\mathbb{Z}^{+n})$$

if and only if $\mathbf{M}_{R_+} = \mathbf{M}_{R_-}$, for all $R \in \mathbf{r}$. $\diamondsuit$

Let us give a specific example of this as follows.

Example 4.3.5.(a) *Let K be the free abelian monoid rank* 2 *with the presentation $\mathcal{P}_K = [y_1, y_2\ ;\ y_1y_2 = y_2y_1]$ as in* (4.2)*, and let A be the one-relator monoid with the presentation $\mathcal{P}_A = [x_1, x_2\ ;\ x_1^2x_2 = x_2x_1]$.*

Let us take two matrices $\mathbf{M}_{x_1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{M}_{x_2} = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$. Thus, since $\mathbf{M}_{x_1}^2\mathbf{M}_{x_2} = \mathbf{M}_{x_2}\mathbf{M}_{x_1}$ then the mapping $x_1 \longmapsto \psi_{\mathbf{M}_{x_1}}$, $x_2 \longmapsto \psi_{\mathbf{M}_{x_2}}$ induces a homomorphism $\theta : A \longrightarrow End(\mathbb{Z}^{+2})$. $\diamondsuit$

By the following example, we define a homomorphism from a finite cyclic monoid to the endomorphism monoid of another finite cyclic monoid.

Example 4.3.6 *Let K and A be two finite cyclic monoids with the presentations*

$$\mathcal{P}_K = \left[y\,;\, y^k = y^l\right], \quad \mathcal{P}_A = \left[x\,;\, x^\mu = x^\lambda\right], \tag{4.10}$$

respectively, where $l < k$ and $\lambda < \mu$ (see Lemma 4.2.7). Let ψ_i $(0 \leqslant i < k)$ be an endomorphism of K (see Example 4.2.15, Case 1). Then we have a mapping

$$x \longrightarrow End(K), \quad x \longmapsto \psi_i.$$

By Lemma 4.2.1, this induces a homomorphism

$$\theta : A \longrightarrow End(K), \quad x \longmapsto \psi_i$$

if and only if

$$\psi_i^\mu = \psi_i^\lambda.$$

Since ψ_i^μ and ψ_i^λ are equal if and only if they agree on the generator y of K, then we must have

$$[y^{i^\mu}] = [y^{i^\lambda}]. \tag{4.11}$$

4.4 The p-Cockcroft property for semi-direct products

4.4.1 The general theorem

Theorem 4.4.1 *Let p be a prime or 0. Then the presentation $\mathcal{P}_D$, as in (4.8), is p-Cockcroft if and only if the following conditions hold.*

(i) $\mathcal{P}_A$ and $\mathcal{P}_K$ are p-Cockcroft,

(ii) $\exp_y(S) \equiv 0\,(mod\ p)$ for all $S \in \mathbf{s}$, $y \in \mathbf{y}$,

(iii) $\exp_{S_0}(\mathbb{B}_{S,x}) \equiv \begin{cases} 1, & S_0 = S \\ 0, & \textit{otherwise} \end{cases}$ *(mod p) for all* $S_0, S \in \mathbf{s}$, $x \in \mathbf{x}$,

(iv) $\exp_S(\mathbb{C}_{y,\theta_R}) \equiv 0$ *(mod p) for all* $S \in \mathbf{s}$, $y \in \mathbf{y}$, $R \in \mathbf{r}$,

(v) $\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) \equiv \exp_{T_{yx}}(\mathbb{A}_{R_-,y})$ *(mod p) for all* $R \in \mathbf{r}$, $y \in \mathbf{y}$ *and* $x \in \mathbf{x}$.

Proof. Since the trivialiser set $\mathbf{X_D}$ contains the trivialiser sets $\mathbf{X_A}$ of $\mathcal{D}(\mathcal{P}_A)$ and $\mathbf{X_K}$ of $\mathcal{D}(\mathcal{P}_K)$ by Theorem 4.3.4, then we must have $\mathcal{P}_A$ and $\mathcal{P}_K$ are p-Cockcroft. This gives the condition (i).

Consider a picture $\mathbb{P}_{S,x}$ ($S \in \mathbf{s}$, $x \in \mathbf{x}$). It contains a single S-disc, some T_{yx} $(y \in \mathbf{y})$ discs and a single $\mathbb{B}_{S,x}^{-1}$ subpicture. First of all, this single S-disc must be balanced by using the subpicture $\mathbb{B}_{S,x}^{-1}$ which contains the remaining $\mathbf{s}$-discs. Thus we must have

$$\exp_{S_0}(\mathbb{B}_{S,x}) \equiv \begin{cases} 1, & S_0 = S \\ 0, & \text{otherwise} \end{cases} \pmod{p},$$

for all $S_0 \in \mathbf{s}$. So the condition (iii) holds. Furthermore, we need to count the number of T_{yx} $(y \in \mathbf{y})$ discs in the $\mathbb{P}_{S,x}$ picture. For a fixed $y \in \mathbf{y}$, the exponent sum of T_{yx} in $\mathbb{P}_{S,x}$ is

$$L_y(S_+) - L_y(S_-) =_{def} \exp_y(S).$$

Thus the condition (ii) must hold.

Consider a picture $\mathbb{P}_{R,y}$ ($R \in \mathbf{r}$, $y \in \mathbf{y}$) which contains the subpictures $\mathbb{A}_{R_+,y}$, $\mathbb{A}_{R_-,y}^{-1}$, $\mathbb{C}_{y,\theta_R}$ and two R-discs. Note that, the exponent sum of the R-discs will be equal to zero for the picture $\mathbb{P}_{R,y}$, that is, we have

$$\exp_R(\mathbb{P}_{R,y}) = 1 - 1 = 0.$$

Let us consider the subpictures $\mathbb{A}_{R_+,y}$ and $\mathbb{A}_{R_-,y}^{-1}$ which consist of only T_{yx} $(x \in \mathbf{x})$ discs. We should note that, T_{yx} $(x \in \mathbf{x})$ discs are only contained in these subpictures, in the picture $\mathbb{P}_{R,y}$. Since the picture $\mathbb{P}_{R,y}$ contains a single subpicture $\mathbb{A}_{R_+,y}$ and single subpicture $\mathbb{A}_{R_-,y}^{-1}$, then we have

$$\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) - \exp_{T_{yx}}(\mathbb{A}_{R_-,y}) = \exp_{T_{yx}}(\mathbb{P}_{R,y}).$$

Thus we must have

$$\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) - \exp_{T_{yx}}(\mathbb{A}_{R_-,y}) \equiv 0 \pmod p,$$

for all $x \in \mathbf{x}$. So, the condition (v) holds. Also, let us consider the subpicture $\mathbb{C}_{y,\theta_R}$ which consist of only the S-discs ($S \in \mathbf{s}$). So, we must have

$$\exp_S(\mathbb{C}_{y,\theta_R}) \equiv 0 \pmod p,$$

for all $S \in \mathbf{s}$, and this gives the condition (iv). $\square$

4.4.2 Direct products

In this section we will give necessary and sufficient conditions for the presentation of the direct product of the monoids A and K to be p-Cockcroft (p a prime or 0).

The direct product corresponds to the case when θ is the trivial homomorphism

$$A \longrightarrow End(K), \quad a \longmapsto id \ (a \in A).$$

So, let us take

$$y\theta_x = y, \tag{4.12}$$

for all $y \in \mathbf{y}$. Then, for $x \in \mathbf{x}$, $y \in \mathbf{y}$, the relator T_{yx} becomes simply

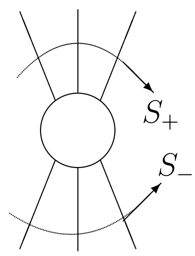
$$T_{yx} : yx = xy.$$

Then the picture $\mathbb{A}_{U,y}$ becomes the picture as shown in Figure 4.4.

By using (4.12), we have

$$\iota(\mathbb{B}_{S,x}) = S_+ \text{ and } \tau(\mathbb{B}_{S,x}) = S_- \ (S \in \mathbf{s},\, x \in \mathbf{x}),$$

for the subpicture $\mathbb{B}_{S,x}$. Then we take $\mathbb{B}_{S,x}$ to be the following form.



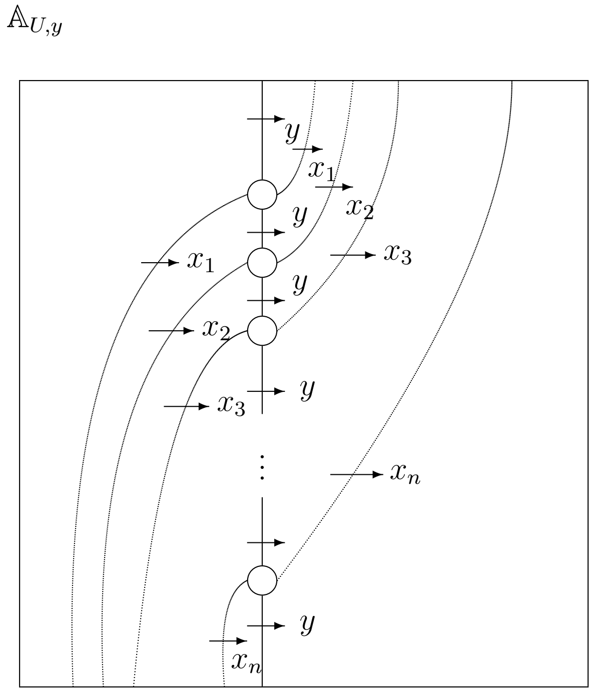


Figure 4.4:

By using (4.12), we have

$$\iota(\mathbb{C}_{y,\theta_R}) = y \text{ and } \tau(\mathbb{C}_{y,\theta_R}) = y \ (R \in \mathbf{r},\ y \in \mathbf{y}),$$

for the subpicture $\mathbb{C}_{y,\theta_R}$. Then $\mathbb{C}_{y,\theta_R}$ can be chosen to consist of a single y-arc and no discs.

Therefore, as a consequence of Theorem 4.4.1, we get the following result.

Theorem 4.4.2 *Suppose that θ is the trivial homomorphism, and let p be a prime or 0. Then the presentation $\mathcal{P}_D$, as in (4.8), is p-Cockcroft if and only if the following conditions hold.*

(a) $\mathcal{P}_A$ *and* $\mathcal{P}_K$ *are p-Cockcroft,*

(b) $\exp_y(S) \equiv 0 \pmod p$ *for all* $S \in \mathbf{s}$, $y \in \mathbf{y}$,

(c) $\exp_x(R) \equiv 0 \pmod p$ *for all* $R \in \mathbf{r}$, $x \in \mathbf{x}$.

Proof. To prove this theorem, let us check the conditions of Theorem 4.4.1 hold.

i) To make (*i*) holds, we definitely need $\mathcal{P}_A$ and $\mathcal{P}_K$ are *p*-Cockcroft. So, this also gives the condition (*a*).

ii) Clearly, the condition (*ii*) gives the condition (*b*).

iii) The condition (*iii*) obviously holds.

iv) The condition (*iv*) clearly holds.

v) It is clear that

$$\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) = L_x(R_+) \text{ and } \exp_{T_{yx}}(\mathbb{A}_{R_-,y}) = L_x(R_-).$$

So, to make (*v*) holds, we need

$$L_x(R_+) - L_x(R_-) \equiv 0 \pmod{p}.$$

That is,

$$\exp_x(R) \equiv 0 \pmod{p}$$

which gives the condition (*c*).

Hence the result. □

Let p be a prime or 0. Let K be the monoid presented by $\mathcal{P}_K = [\mathbf{y}\ ;\ \mathbf{s}]$, and let A be an infinite cyclic monoid generated by x. Then, a presentation for the monoid $K \times \mathbb{Z}^+$ can be given by

$$\mathcal{P}_{K\times\mathbb{Z}^+} = [\mathbf{y},\, x\ ;\ \mathbf{s},\, yx = xy\ \ (y \in \mathbf{y})]\,. \tag{4.13}$$

As a consequence of Theorem 4.4.2 (so that Theorem 4.4.1), we have

Corollary 4.4.3 *Let p be a prime or 0. The presentation $\mathcal{P}_{K\times\mathbb{Z}^+}$, as in (4.13), is p-Cockcroft if and only if*

(a') *$\mathcal{P}_K$ is p-Cockcroft,*

(b') *$\exp_y(S) \equiv 0$ (mod p) for all $y \in \mathbf{y}$ and $S \in \mathbf{s}$.*

Proof. The proof is an easy application of the proof of Theorem 4.4.2. To make (a) hold, we certainly need $\mathcal{P}_K$ is p-Cockcroft. Notice that, $\mathcal{P}_A$ is aspherical, hence Cockcroft. So, these give the condition (a'). Clearly, (b) gives (b'), and the condition (c) is vacuous. □

Example 4.4.4 *As an example of Corollary* 4.4.3, *let us prove by induction on* n *that the presentation* $\mathcal{P}$, *as in* (4.2), *presents the monoid* $\mathbb{Z}^{+n}$, *and is Cockcroft.*

- *Let* $n = 1$. *Then, we get* $\mathbb{Z}^+$ *which is infinite cyclic monoid with a presentation*

$$\mathcal{P}_1 = [y_1 \ ; \quad] .$$

Then, $\mathcal{P}_1$ *is aspherical, hence Cockcroft.*

- *Let us assume that*

$$\mathcal{P}_{n-1} = [y_1, \, y_2 \cdots, \, y_{n-1} \ ; \ y_i y_j = y_j y_i \ (1 \leqslant i < j \leqslant n-1)]$$

is a presentation of $\mathbb{Z}^{+n-1}$ *and that it is Cockcroft. Let* $\mathbf{y}$ *be the set of generators* $y_1, \cdots, y_{n-1}$, *let* $\mathbf{s}$ *be the set of relators* $y_i y_j = y_j y_i$ $(1 \leqslant i < j \leqslant n-1)$, *and let* x *be the generator* y_n. *Then the set of relators* $y_i y_n = y_n y_i$ $(1 \leqslant i \leqslant n-1)$ *becomes the set of relators* $\mathbf{t}$. *Thus we have a presentation*

$$\begin{aligned} \mathcal{P}_n \;\; = \;\; & [y_1, \, y_2 \cdots, \, y_n \ ; \ y_i y_j = y_j y_i \ (1 \leqslant i < j \leqslant n-1) \\ & y_i y_n = y_n y_i \ (1 \leqslant i \leqslant n-1)] \end{aligned}$$

of the monoid $\mathbb{Z}^{+n} = \mathbb{Z}^{+n-1} \times \mathbb{Z}^+$, *as in* (4.13). *Notice that the presentation* $\mathcal{P}$, *as in* (4.2) *and* $\mathcal{P}_n$ *are equivalent. To establish the Cockcroft property of* $\mathcal{P}_n$, *let us use Corollary* 4.4.3. *By inductive hypothesis* $\mathcal{P}_{n-1}$ *is Cockcroft, so the condition* (a') *holds. Also, for all* $S \in \mathbf{s}$, $y \in \mathbf{y}$, $\exp_y(S) = 1 - 1 = 0$ *which gives the condition* (b'). *Thus,* $\mathcal{P}_n$ *is Cockcroft, as required.*

4.4.3 Semi-direct products of finite cyclic monoids

In this section we will give necessary and sufficient conditions for the presentation of the semi-direct product of two finite cyclic monoids to be p-Cockcroft (p a prime).

Let K and A be two finite cyclic monoids with the presentations $\mathcal{P}_K$ and $\mathcal{P}_A$, respectively as in (4.10). Suppose that

$$[y^{i^\mu}] = [y^{i^\lambda}].$$

Then the mapping $x \longmapsto \psi_i$ induces a homomorphism $\theta : A \longrightarrow End(K)$ (see Example 4.3.6).

Now, by Theorem 4.3.2, we have a presentation

$$\mathcal{P}_D = [y,\, x\ ;\ S,\, R,\, T_{yx}]\,, \tag{4.14}$$

for the monoid $D = K \rtimes_\theta A$, where

$$S : y^k = y^l,\ \ R : x^\mu = x^\lambda \ \text{ and } \ T_{yx} : yx = xy^i.$$

We have the picture $\mathbb{B}_{S,x}$ as in Figure 4.5, and then $\exp_S(\mathbb{B}_{S,x}) = i$.

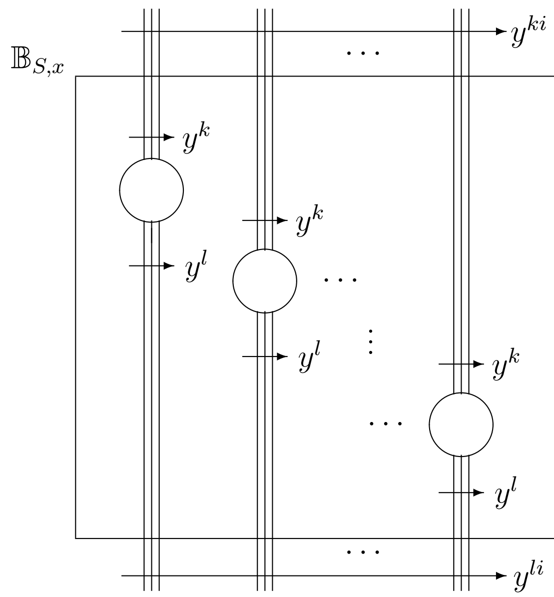


Figure 4.5:

By the assumption, since (4.11) holds then, by Lemma 4.2.11, there is a monoid picture $\mathbb{C}_{y,\theta_R}$ with

$$\iota(\mathbb{C}_{y,\theta_R}) = y^{i^\mu}, \quad \tau(\mathbb{C}_{y,\theta_R}) = y^{i^\lambda}$$

and

$$\exp_S(\mathbb{C}_{y,\theta_R}) = \frac{i^\mu - i^\lambda}{k-l}.$$

Also, we have the picture $\mathbb{A}_{R_+,y}$ (and similarly $\mathbb{A}_{R_-,y}$) as in Figure 4.6. It is clear

$\mathbb{A}_{R_+,y}$

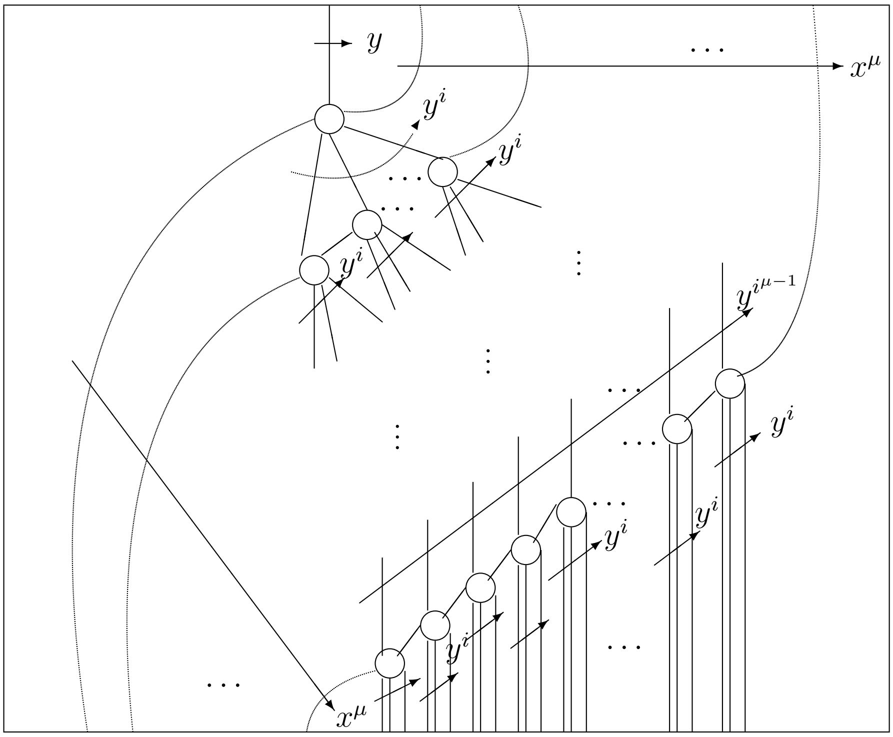


Figure 4.6:

that

$$\exp_{T_{yx}}(\mathbb{A}_{R_+,y}) = 1 + i + i^2 + \cdots + i^{\mu-1} = \frac{i^\mu - 1}{i-1},$$

and

$$\exp_{T_{yx}}(\mathbb{A}_{R_-,y}) = 1 + i + i^2 + \cdots + i^{\lambda-1} = \frac{i^\lambda - 1}{i-1}.$$

Let

$$m = k - l\ ,\ n = i - 1\ ,\ t = i^\mu - i^\lambda.$$

As a consequence of Theorem 4.4.1, we have the following result.

Theorem 4.4.5 *Let p be a prime. Suppose that $K \rtimes_\theta A$ is a monoid with the associated monoid presentation $\mathcal{P}_D$, as in* (4.14)*. Then $\mathcal{P}_D$ is p-Cockcroft if and only if*

$$p \mid m,\ p \mid n,\ p \mid \frac{t}{m},\ p \mid \frac{t}{n}.$$

Proof. We will prove it by checking the conditions of Theorem 4.4.1 hold.

(i) By Lemma 4.2.13, trivialiser sets $\mathbf{X_K}$ and $\mathbf{X_A}$ of the Squier complexes $\mathcal{D}(\mathcal{P}_K)$ and $\mathcal{D}(\mathcal{P}_A)$ respectively, can be given as in Figure 4.1. Thus, it can be seen that $\mathcal{P}_K$ and $\mathcal{P}_A$ are p-Cockcroft (in fact Cockcroft), and then the condition (i) holds.

ii) $\exp_y(S) = k - l$ so for (ii) to hold, we must have $p \mid k - l$.

iii) To make (iii) hold, we need $i \equiv 1 \pmod p$, so that $p \mid i - 1$.

iv) For the subpicture $\mathbb{C}_{y,\theta_R}$, we must have

$$p \mid \frac{i^\mu - i^\lambda}{k - l},$$

to make (iv) hold.

v) Also, to make (v) hold, we need

$$\frac{i^\mu - 1}{i - 1} \equiv \frac{i^\lambda - 1}{i - 1} \pmod p,$$

by using the subpictures $\mathbb{A}_{R_+,y}$ and $\mathbb{A}_{R_-,y}$. That is,

$$\frac{i^\mu - i^\lambda}{i - 1} \equiv 0 \pmod p.$$

Hence the result. □

We remark that as a consequence of Theorem 4.4.1, one can say that the monoid presentation $\mathcal{P}_D$, as in (4.14), is Cockcroft if and only if $\mu = \lambda$, $k = l$ and $i = 1$. However, since we require $l < k$, $\lambda < \mu$ then this presentation can never be Cockcroft.

Example 4.4.6 *Let $k = 10$, $l = 6$, $\mu = 4$, $\lambda = 2$ and $i = 3$. Then we get*

$$m = 4,\ n = 2,\ t = 3^4 - 3^2 = 72,\ \frac{t}{m} = 18,\ \frac{t}{n} = 36.$$

Hence $p = 2$ divides these all values, and then by Theorem 4.4.5, $\mathcal{P}_D$ is 2-Cockcroft. Similarly, by chosing $k = 6$, $l = 2$, $\mu = 5$, $\lambda = 3$ and $i = 3$, we get

$$m = 4,\ n = 2,\ t = 3^5 - 3^4 = 216,\ \frac{t}{m} = 54,\ \frac{t}{n} = 108,$$

then again $\mathcal{P}_D$ is 2-Cockcroft. $\Diamond$

Example 4.4.7 *Let p be any prime, and let*

$$i = p + 1,\ \ l = 1,\ \ k = (p+1)(\frac{(p+1)^p - 1}{p}) + 1,\ \ \lambda = 1,\ \ \mu = p + 1.$$

Then,

$$m = (p+1)(\frac{(p+1)^p - 1}{p}),\ \ n = p,\ \ t = (p+1)^{p+1} - (p+1)^1.$$

Since p divides m, n, $\frac{t}{m}$ and $\frac{t}{n}$ then, by Theorem 4.4.5, $\mathcal{P}_D$ is p-Cockcroft. $\Diamond$

Chapter 5

Minimal presentations of semi-direct products of some monoids

5.1 Introduction

In this chapter, as an application of the previous chapter, we begin by giving necessary and sufficient conditions for a semi-direct product of a one-relator monoid by an infinite cyclic monoid to be p-Cockcroft, for any prime p or 0, and then we give some applications of this to semi-direct products of the free abelian monoid of rank 2 by an infinite cyclic monoid, and semi-direct products of some particular one-relator monoids by an infinite cyclic monoid.

Following this, we introduce our main result of this chapter which gives sufficient conditions for the presentation of a semi-direct product of a one-relator monoid by an infinite cyclic monoid to be minimal but not efficient, and then we give some applications of this.

5.2 Semi-direct products of one-relator monoids by infinite cyclic monoids

Let K be a one-relator monoid with presentation $\mathcal{P}_K = [\mathbf{y}\ ;\ S_+ = S_-]$, and let A be the infinite cyclic monoid with presentation $\mathcal{P}_A = [x\ ;\ ]$. Let ψ be an endomorphism of K. Then by Section 4.3.5, the mapping $x \longmapsto \psi$ induces a homomorphism

$$\theta : A \longrightarrow End(K),$$

and we can form the semi-direct product $D = K \rtimes_\theta A$. This will have a presentation

$$\mathcal{P}_D = [\mathbf{y},\, x\ ;\ S_+ = S_-,\ \mathbf{t}]\,, \tag{5.1}$$

where $\mathbf{t}$ is the set of relators T_{yx} $(y \in \mathbf{y})$. Notice that, since $\mathcal{P}_A = [x\ ;\ ]$ is aspherical then $\mathbf{X_A} = \emptyset$. Also, for the relator S, let us assume that $\iota(S_+) \neq \iota(S_-)$ (or $\tau(S_+) \neq \tau(S_-)$). So, by [34], $\mathcal{P}_K$ is aspherical, so $\mathbf{X_K} = \emptyset$. Moreover, since $\mathbf{r} = \emptyset$ then $\mathbf{C_2} = \emptyset$. Therefore $\mathbf{X_D} = \mathbf{C_1}$. Note that we have a single $\mathbb{P}_{S,x}$ picture, as in Figure 5.1, in the set $\mathbf{C_1}$ since K is a one-relator monoid.

5.2.1 The p-Cockcroft property

Theorem 5.2.1 *Let p be a prime or 0, and let K be a one-relator monoid, with relator S say. Suppose that $\iota(S_+) \neq \iota(S_-)$ (or $\tau(S_+) \neq \tau(S_-)$). Let D be a semi-direct product of K by an infinite cyclic monoid A with associated presentation $\mathcal{P}_D$, as in (5.1). Then $\mathcal{P}_D$ is p-Cockcroft if and only if*

(a) $\exp_y(S) \equiv 0\ (mod\ p)$ *for all* $y \in \mathbf{y}$,

(b) $\exp_S(\mathbb{B}_{S,x}) \equiv 1\ (mod\ p)$.

Proof. It is an easy application of the proof of Theorem 4.4.1.

Since $\mathcal{P}_A$ and $\mathcal{P}_K$ are aspherical and $\mathbf{C_2} = \emptyset$ then the conditions (i), (iv) and (v) of Theorem 4.4.1 are trivial. On the other hand, the condition (ii) gives (a) and the condition (iii) gives (b).

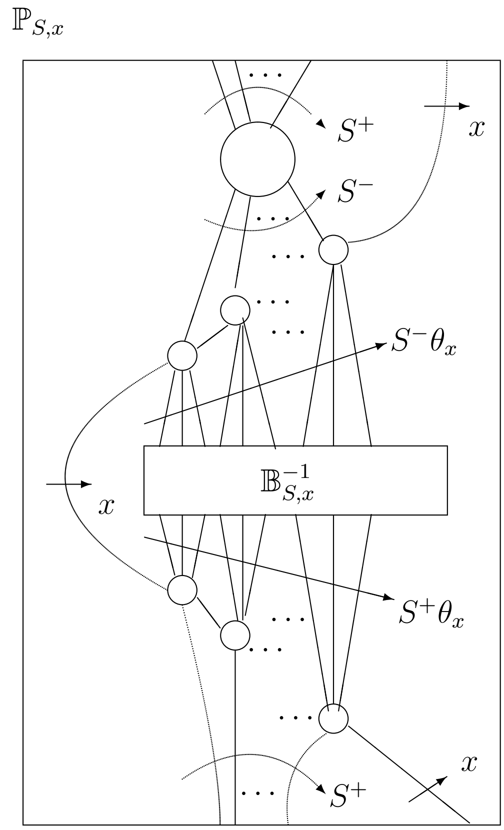


Figure 5.1:

Hence the result. □

Example 5.2.2 *Let K be the free abelian monoid of rank 2, presented by*

$$\mathcal{P}_K = [y_1,\, y_2\ ;\ y_1y_2 = y_2y_1]\,,$$

and let ψ be the endomorphism $\psi_{\mathbf{M}}$ where $\mathbf{M}$ is the matrix $\begin{bmatrix} \alpha & \alpha' \\ \beta & \beta' \end{bmatrix}$ *($\alpha,\, \alpha',\, \beta,\, \beta' \in \mathbb{Z}^+$), given by*

$$[y_1] \longmapsto [y_1^{\alpha} y_2^{\alpha'}] \ \text{ and } \ [y_2] \longmapsto [y_1^{\beta} y_2^{\beta'}]$$

(see Examples 4.3.5 and 4.3.5.(a)).

By Theorem 4.3.2, we have the presentation

$$\mathcal{P}_D = [y_1,\, y_2,\, x\ ;\ S,\, T_{y_1x},\, T_{y_2x}]\,, \tag{5.2}$$

for the monoid $D = K \rtimes_\theta A$, *where*

$$S : y_1 y_2 = y_2 y_1,\ T_{y_1 x} : y_1 x = x y_1^{\alpha} y_2^{\alpha'} \ \ and\ \ T_{y_2 x} : y_2 x = x y_1^{\beta} y_2^{\beta'},$$

respectively. Note that the picture $\mathbb{B}_{S,x}$ *can be given by Figure* 5.2. ♢

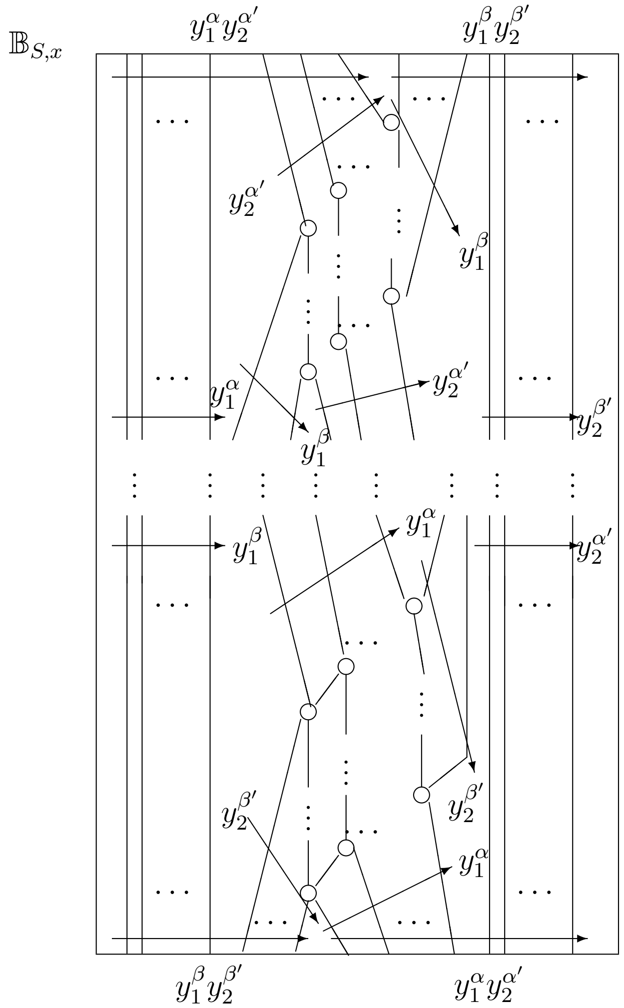


Figure 5.2:

Now, by considering the picture $\mathbb{B}_{S,x}$ as in Figure 5.2, we prove the following equality.

Lemma 5.2.3

$$\exp_S(\mathbb{B}_{S,x}) = det\mathbf{M}.$$

Proof. We have $\alpha\beta'$-times positive and $\alpha'\beta$-times negative S-discs, in $\mathbb{B}_{S,x}$. So that

$$\begin{aligned} \exp_S(\mathbb{B}_{S,x}) &= \alpha\beta' - \alpha'\beta, \\ &= \det \mathbf{M}, \end{aligned}$$

as required. □

As a consequence of Theorem 5.2.1, we have

Corollary 5.2.4 *Let p be a prime or 0. Let $\mathcal{P}_D$ be as in (5.2). Then $\mathcal{P}_D$ is p-Cockcroft if and only if*

$$\det \mathbf{M} \equiv 1 \pmod{p}.$$

Proof. Let us check the conditions of Theorem 5.2.1 hold.

Since $\exp_{y_1}(S) = 0 = \exp_{y_2}(S)$ then (a) holds. Also, by Lemma 5.2.3, (b) holds if and only if $\det \mathbf{M} \equiv 1 \pmod{p}$. □

Example 5.2.5 *Let K be the one-relator monoid with the presentation*

$$\mathcal{P}_K = [y_1,\, y_2 \;;\; S]\,,$$

where $S : y_1y_2y_1 = y_2y_1^k$, and let ψ_x be the endomorphism given by

$$[y_1] \longmapsto [y_1^i] \;\; \textit{and} \;\; [y_2] \longmapsto [y_2],$$

where $i \in \mathbb{Z}^+$ (see Example 4.2.16.(a)). By Theorem 4.3.2, we have the presentation

$$\mathcal{P}_D = \left[y_1,\, y_2,\, x \;;\; S,\, y_1x = xy_1^i,\, y_2x = xy_2\right] \tag{5.3}$$

for the monoid $D = K \rtimes_\theta A$. The picture $\mathbb{B}_{S,x}$ can be given by Figure 5.3. ◊

We get the following result for the above example, as a consequence of Theorem 5.2.1.

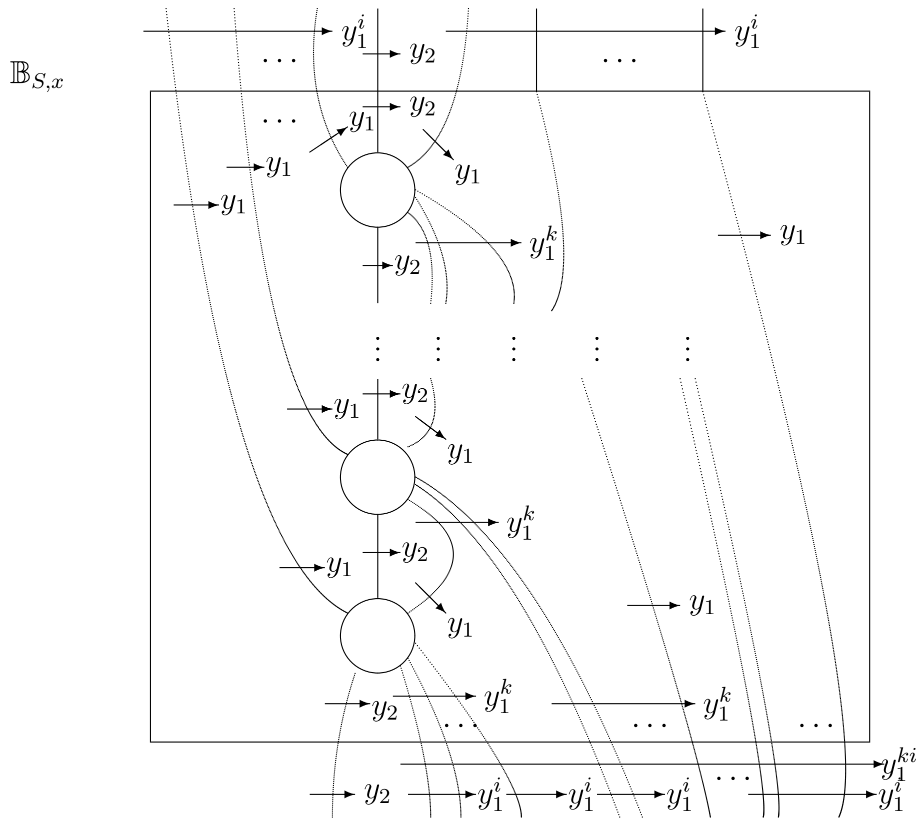


Figure 5.3:

Corollary 5.2.6 *Let p be a prime or 0. Let $\mathcal{P}_D$ be as in (5.3). Then $\mathcal{P}_D$ is p-Cockcroft if and only if*

a') $k \equiv 2$ (mod p)

and

b') $i \equiv 1$ (mod p).

Proof. Let us check the conditions of Theorem 5.2.1 hold.

It is clear that $\exp_{y_1}(S) = 2 - k$ and $\exp_{y_2}(S) = 1 - 1 = 0$. Then to make the condition (a) hold, we must have $k - 2 \equiv 0 \pmod p$ which gives a'). Also, since $\exp_S(\mathbb{B}_{S,x}) = i$ then the condition (b) gives b').

Hence the result. □

Example 5.2.5 (continued) *One can choose $k = 2$ and $i = 3$ then $\mathcal{P}_D$ is 2-Cockcroft, or $k = 5$ and $i = 4$ then $\mathcal{P}_D$ is 3-Cockcroft.* ◇

Remark 5.2.7 *It is easy to see that if $k = 2$ and $i = 1$ then $\mathcal{P}_D$ is* 0*-Cockcroft. But the condition $i = 1$ implies that ψ_x is the identity map and so θ is the trivial homomorphism, as in* (4.12)*. Then the presentation $\mathcal{P}_D$ becomes a presentation, as in* (4.13)*, of the direct product $K \times \mathbb{Z}^+$. Thus, by Corollary* 4.4.3*, we can see directly $\mathcal{P}_D$ is* 0*-Cockcroft when $k = 2$ and $i = 1$.*

Example 5.2.8 *Let K be given by the presentation $\mathcal{P}_K = [y_1, y_2\,;\, S]$, where $S : y_1^k y_2 = y_2 y_1^k$, and let ψ_x be the endomorphism given by*

$$[y_1] \longmapsto [y_1^i] \text{ and } [y_2] \longmapsto [y_2^j],$$

where $i, j \in \mathbb{Z}^+$ (see Example 4.2.16.(*b*))*. By Theorem* 4.3.2*, we have a presentation*

$$\mathcal{P}_D = \left[y_1, y_2, x\,;\, S, y_1 x = x y_1^i, y_2 x = x y_2^j\right] \tag{5.4}$$

for the monoid D. For this example, the picture $\mathbb{B}_{S,x}$ can be given by Figure 5.4*.* $\diamondsuit$

We then get the following, as a consequence of Theorem 5.2.1.

Corollary 5.2.9 *Let p be a prime or* 0*. Let $\mathcal{P}_D$ be as in* (5.4)*. Then $\mathcal{P}_D$ is p-Cockcroft if and only if*

$$ij \equiv 1 \ (mod\ p).$$

Proof. Again, let us check the conditions of Theorem 5.2.1 hold.

Since $\exp_{y_1}(S) = k - k = 0$ and $\exp_{y_2}(S) = 1 - 1 = 0$ then the condition (a) holds. Also, since $\exp_S(\mathbb{B}_{S,x}) = ij$ then to make the condition (b) hold, we must have $ij \equiv 1$ (mod p) which gives the condition of the above corollary, as required. $\square$

Example 5.2.8 (continued) *One can choose $i = 3$ and $j = 1$ then $\mathcal{P}_D$ is* 2*-Cockcroft, or $i = j = 2$ then $\mathcal{P}_D$ is* 3*-Cockcroft.* $\diamondsuit$

Remark 5.2.10 *It is clear that if $i = j = 1$ then $\mathcal{P}_D$ is* 0*-Cockcroft. But as we said in Remark* 5.2.7*, the presentation $\mathcal{P}_D$ becomes a presentation, as in* (4.13)*, of the direct product $K \times \mathbb{Z}^+$ when $i = j = 1$ holds. Then, by Corollary* 4.4.3*, one can say directly the presentation $\mathcal{P}_D$ is* 0*-Cockcroft.*

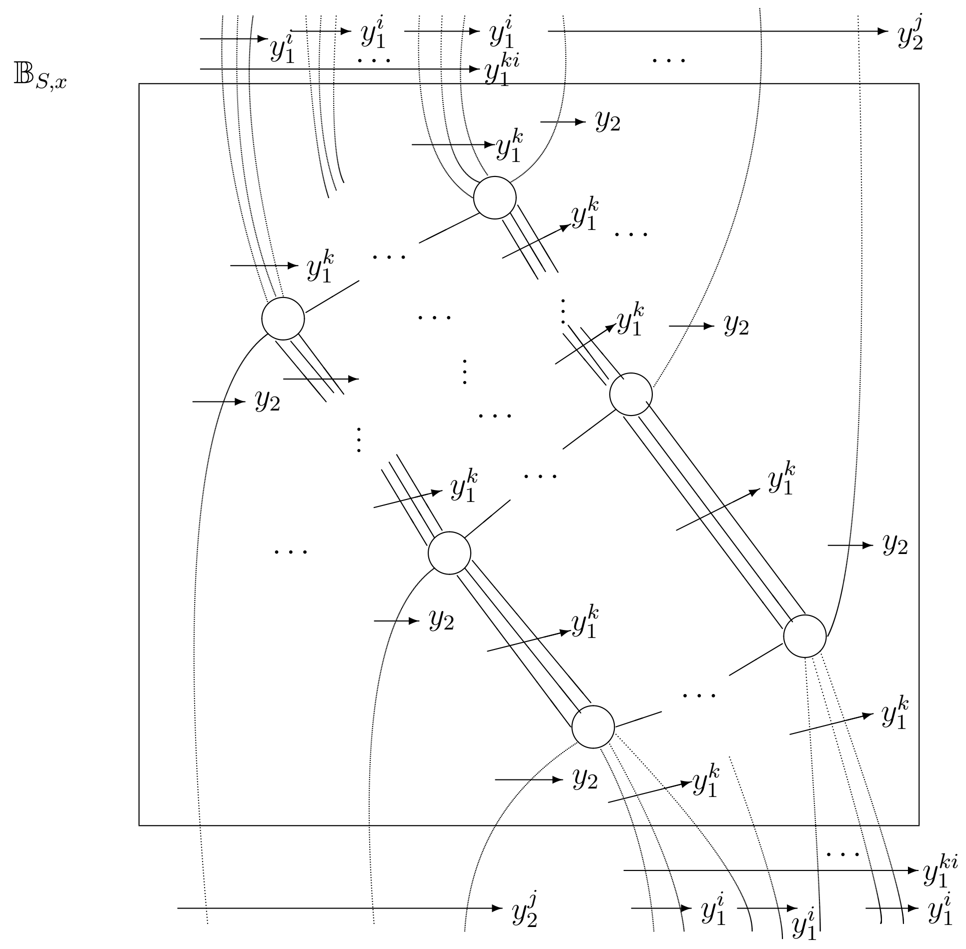


Figure 5.4:

A similar example can be given as follows.

Example 5.2.11 *Let K be given by the presentation $\mathcal{P}_K = [y_1, y_2\ ;\ S]$, where S : $y_1y_2 = y_2y_1^k$, and let ψ_x be the endomorphism given by*

$$[y_1] \longmapsto [y_1^i] \ \ \textit{and} \ \ [y_2] \longmapsto [y_2],$$

where $i \in \mathbb{Z}^+$ (see Example 4.2.16.(c)). By Theorem 4.3.2, we have a presentation

$$\mathcal{P}_D = \left[y_1,\, y_2,\, x\ ;\ S,\, y_1x = xy_1^i,\, y_2x = xy_2\right] \tag{5.5}$$

for the monoid D. Also, the picture $\mathbb{B}_{S,x}$ can be given by Figure 5.5. $\Diamond$

Thus, as a consequence of Theorem 5.2.1, we get

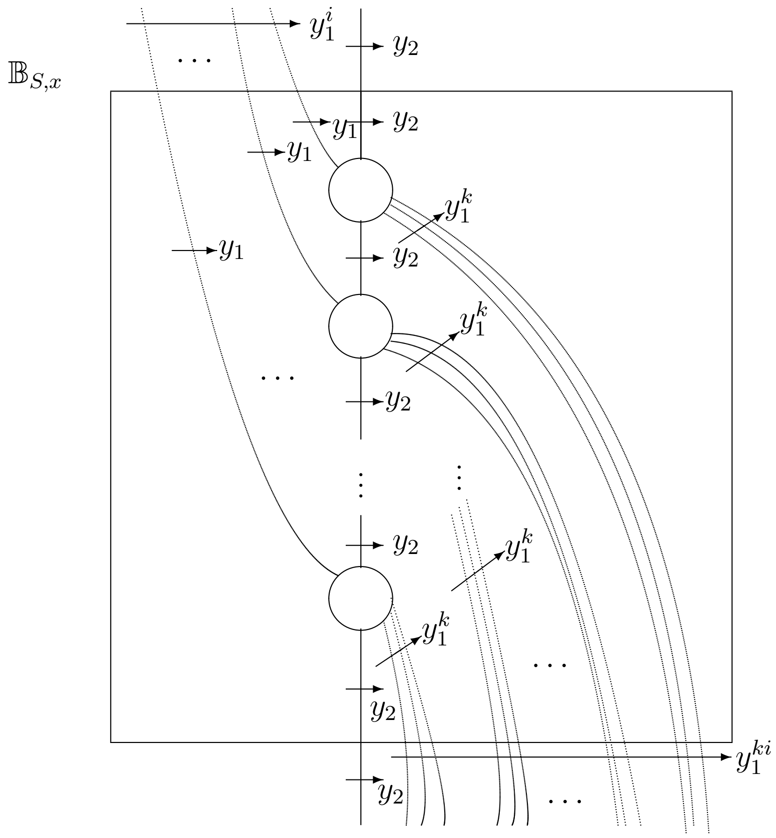


Figure 5.5:

Corollary 5.2.12 *Let p be a prime or 0. Let $\mathcal{P}_D$ be as in (5.5). Then $\mathcal{P}_D$ is p-Cockcroft if and only if*

a') $k \equiv 1$ (mod p),

b') $i \equiv 1$ (mod p).

Proof. Again, let us check the conditions of Theorem 5.2.1 hold.

Clearly $\exp_{y_1}(S) = 1 - k$ and $\exp_{y_2}(S) = 1 - 1 = 0$, so to make the condition (a) hold, we must have $k - 1 \equiv 0$ (mod p) which gives a'). Also, since $\exp_S(\mathbb{B}_{S,x}) = i$ then the condition (b) gives b').

Hence the result. $\square$

Example 5.2.11 (continued) *One can choose $i = 5$ and $k = 7$ then $\mathcal{P}_D$ is 2-Cockcroft, or $i = k = 4$ then $\mathcal{P}_D$ is 3-Cockcroft.* $\diamondsuit$

Remark 5.2.13 *Clearly if* $i = k = 1$ *then* $\mathcal{P}_D$ *is* 0*-Cockcroft. But as we said in Remarks* 5.2.7 *and* 5.2.10*, the presentation* $\mathcal{P}_D$ *becomes a presentation, as in* (4.13)*, of the direct product* $K \times \mathbb{Z}^+$ *when* $i = k = 1$ *holds. Then, by Corollary* 4.4.3*, one can say directly the presentation* $\mathcal{P}_D$ *is* 0*-Cockcroft.*

5.3 Some minimal but inefficient presentations

As we mentioned in Chapter 1, a presentation is efficient if and only if it is p-Cockcroft, for some prime p. It follows from Theorems 5.2.1 and 4.4.1 that the presentation $\mathcal{P}_D$, as in (5.1), is efficient if and only if there is a prime p such that

- $\exp_y(S) \equiv 0 \pmod{p}$ for all $y \in \mathbf{y}$,
- $\exp_S(\mathbb{B}_{S,x}) \equiv 1 \pmod{p}$,

in other words, if and only if

$$hcf(\exp_y(S)\ \ (y \in \mathbf{y}),\ \exp_S(\mathbb{B}_{S,x}) - 1) \neq 1.$$

In particular, $\mathcal{P}_D$ is not efficient if

$$\exp_S(\mathbb{B}_{S,x}) = 0 \text{ or } 2.$$

Let $d = hcf(\exp_y(S)\ \ (y \in \mathbf{y}))$. The value of d will be taken to be 0 if all exponent sums are 0 in $hcf(\exp_y(S) : y \in \mathbf{y})$.

Our main result of this chapter is the following.

Theorem 5.3.1 *The presentation* $\mathcal{P}_D$*, as in* (5.1)*, is* **minimal** (*but not efficient*) *if*

$$d \neq 2^n \text{ and } \exp_S(\mathbb{B}_{S,x}) = 2,$$

for any $n \in \mathbb{Z}^+$*.*

To prove this theorem, we need the following material.

Let us consider the picture $\mathbb{P}_{S,x}$, as in Figure 5.1.

Recall that, for a fixed $y \in \mathbf{y}$, $\dfrac{\partial}{\partial y}$ denotes Fox derivation with respect to y, and $\dfrac{\partial^D}{\partial y}$ is the composition

$$\mathbb{Z}F(\mathbf{y}) \xrightarrow{\frac{\partial}{\partial y}} \mathbb{Z}F(\mathbf{y}) \longrightarrow \mathbb{Z}D,$$

where $F(\mathbf{y})$ is the free monoid on $\mathbf{y}$. Moreover, for the relator S, we define $\dfrac{\partial^D S}{\partial y}$ to be

$$\frac{\partial^D S_+}{\partial y} - \frac{\partial^D S_-}{\partial y}.$$

For a fixed $y \in \mathbf{y}$, let us write

$$S_+ = U_0 y U_1 y \cdots U_{r-1} y U_r \text{ and } S_- = V_0 y V_1 y \cdots V_{k-1} y V_k,$$

where each U_i $(1 \leqslant i \leqslant r)$ and V_j $(1 \leqslant j \leqslant k)$ is a word on $\mathbf{y} - \{y\}$. Then, for this particular y, the left evaluations of the positive atomic pictures in $\mathbb{P}_{S,x}$ (see Chapter 1) containing a T_{yx} disc are

$$\overline{U_0} e_{T_{yx}}, \overline{U_0 y U_1} e_{T_{yx}}, \cdots, \overline{U_0 y \cdots U_{r-1}} e_{T_{yx}},$$

and the left evaluations of the negative atomic pictures in $\mathbb{P}_{S,x}$ containing a T_{yx} disc are

$$-\overline{V_0} e_{T_{yx}}, -\overline{V_0 y V_1} e_{T_{yx}}, \cdots, -\overline{V_0 y \cdots V_{r-1}} e_{T_{yx}}.$$

Hence, for a fixed y, the coefficient of $e_{T_{yx}}$ in $eval^{(l)}(\mathbb{P}_{S,x})$ is

$$\overline{U_0} + \overline{U_0 y U_1} + \cdots + \overline{U_0 y \cdots U_{r-1}} - (\overline{V_0} + \overline{V_0 y V_1} + \cdots + \overline{V_0 y \cdots V_{r-1}}) = \frac{\partial^D S}{\partial y}. \quad (5.6)$$

Lemma 5.3.2 *The second Fox ideal $I_2^{(l)}(\mathcal{P}_D)$ of $\mathcal{P}_D$ is generated by the elements*

$$1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x})) \text{ and } \frac{\partial^D S}{\partial y} \ (y \in \mathbf{y}).$$

Proof. Since $\mathcal{D}(\mathcal{P}_D)$ has a trivialiser $\mathbf{X_D}$ consisting of the single picture $\mathbb{P}_{S,x}$, we need to consider $eval^{(l)}(\mathbb{P}_{S,x})$. We have

$$eval^{(l)}(\mathbb{P}_{S,x}) = \lambda_{\mathbb{P}_{S,x},S} e_S + \sum_{y \in \mathbf{y}} \lambda_{\mathbb{P}_{S,x},T_{yx}} e_{T_{yx}},$$

where

$$\lambda_{\mathbb{P}_{S,x},S} = (1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))),$$
$$\lambda_{\mathbb{P}_{S,x},T_{yx}} = \frac{\partial^D S}{\partial y} \ \ (y \in \mathbf{y}) \ \text{ by (5.6).}$$

Thus, by Remark 1.3.4, we get the result. □

Lemma 5.3.3

$$aug(eval^{(l)}(\mathbb{B}_{S,x})) = \exp_S(\mathbb{B}_{S,x}).$$

Proof. We can write

$$eval^{(l)}(\mathbb{B}_{S,x}) = \varepsilon_1\overline{W_1}e_S + \varepsilon_2\overline{W_2}e_S + \cdots + \varepsilon_n\overline{W_n}e_S,$$

where $\varepsilon_i = \pm 1$ and the W_i's are certain words on $\mathbf{y}$ $(1 \leqslant i \leqslant n)$. In the above expression, each term $\varepsilon_i\overline{W_i}e_S$ corresponds to a single S-disc. Also, the value of each ε_i gives the sign of this single S-disc. Therefore the sum of the ε_i's, that is, $aug(eval^{(l)}(\mathbb{B}_{S,x}))$ must give the exponent sum of the S-discs in the picture $\mathbb{B}_{S,x}$, as required. □

The following lemma is a special case of Lemma 1.3.1 on Fox derivations (see Section 1.3.1).

Lemma 5.3.4

$$aug(\frac{\partial^D S}{\partial y}) = \exp_y(S) \quad (y \in \mathbf{y}).$$

Now we can prove Theorem 5.3.1, as follows.

Suppose that d is not equal to 2^n $(n \in \mathbb{Z}^+)$. Let

$$\mathbb{Z}_d = \begin{cases} \mathbb{Z} & d = 0 \\ \mathbb{Z} \pmod{d} & d \neq 0 \end{cases}.$$

Suppose also that $\exp_S(\mathbb{B}_{S,x}) = 2$.

Let us consider the homomorphism from D onto the infinite cyclic monoid generated by x, defined by

$$y \longmapsto 1 \ (y \in \mathbf{y}), \ x \longmapsto x.$$

This induces a ring homomorphism

$$\gamma : \mathbb{Z}D \longrightarrow \mathbb{Z}[x].$$

Note that the restriction of γ to the subring $\mathbb{Z}K$ of $\mathbb{Z}D$ is just the augmentation map

$$aug : \mathbb{Z}K \longrightarrow \mathbb{Z}.$$

Thus, by Lemmas 5.3.3 and 5.3.4, the image of $I_2^{(l)}(\mathcal{P}_D)$ under γ is the ideal of $\mathbb{Z}[x]$ generated by

$$1 - \bar{x}(\exp_S(\mathbb{B}_{S,x})) = 1 - 2\bar{x},\ \exp_y(S)\ \ (y \in \mathbf{y}).$$

Let η be the composition of γ and the mapping

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}_d[x], \quad x \longmapsto x,\ n \longmapsto \overline{n}\ (n \in \mathbb{Z}),$$

where $\overline{n}$ is $n \pmod{d}$. Then, since $\exp_y(S) \equiv 0 \pmod{d}\ \ (y \in \mathbf{y})$, we get

$$\begin{aligned} \eta(I_2^{(l)}(\mathcal{P}_D)) &= < 1 - \overline{2}\bar{x} > \\ &= I, \text{ say.} \end{aligned}$$

Lemma 5.3.5

$$I \neq \mathbb{Z}_d[x].$$

Proof. For simplicity, we shall replace $\bar{x}$ by x and $\overline{2}$ by 2. Thus we have $I =< 1-2x >$. Then

$$< 1 - 2x >= \{p(x)(1 - 2x) : p(x) \in \mathbb{Z}_d[x]\}. \tag{5.7}$$

Suppose that $< 1 - 2x >= \mathbb{Z}_d[x]$ or equivalently, $1 \in I$. So, $1 = (1 - 2x)p(x)$ for some polynomial $p(x) \in \mathbb{Z}_d[x]$. Write $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r$ where $a_0, a_1, a_2, \cdots, a_r \in \mathbb{Z}_d$. Then

$$1 = a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + \cdots + (a_r - 2a_{r-1})x^r - 2a_rx^{r+1}.$$

Thus $a_0 - 1 \equiv 0 \pmod{d}$, $a_1 - 2a_0 \equiv 0 \pmod{d}$, $\cdots$, $a_r - 2a_{r-1} \equiv 0 \pmod{d}$ and $-2a_r \equiv 0 \pmod{d}$. Since $d \neq 1, 2^n$, we can choose an **odd** prime p such that $p \mid d$.

So, $p \mid -a_r$ (since p is odd then p does not divide 2, but we know that $p \mid d$ and $-2a_r \equiv 0 \pmod{d}$ then $p \mid -a_r$). Also, since $p \mid d$, $a_r - 2a_{r-1} \equiv 0 \pmod{d}$ then $p \mid -2a_{r-1} \Rightarrow p \mid -a_{r-1}$. Similarly, since $p \mid d$ and $a_{r-1} - 2a_{r-2} \equiv 0 \pmod{d}$ then $p \mid -2a_{r-2} \Rightarrow p \mid -a_{r-2}$. By iterating this procedure, we get $p \mid a_0$. Thus, since $p \mid d$ and $a_0 - 1 \equiv 0 \pmod{d}$ then $p \mid 1$. But it is a contradiction. Therefore $< 1 - 2x > \neq \mathbb{Z}_d[x]$, as required. $\square$

Let ψ be the composition

$$\mathbb{Z}D \xrightarrow{\eta} \mathbb{Z}_d[x] \xrightarrow{\phi} \mathbb{Z}_d[x]/I,$$

where ϕ is the natural epimorphism. Then ψ sends $I_2^{(l)}(\mathcal{P}_D)$ to 0, and $\psi(1) = 1$. In other words, the images of the generators of $I_2(\mathcal{P}_D)$ are all 0 under ψ. That is,

$$\begin{aligned} \psi(1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) &= \phi\eta(1 - \bar{x}(eval^{(l)}(\mathbb{B}_{S,x}))) \\ &= \phi(1 - \bar{x}(\overline{\exp_S(\mathbb{B}_{S,x})}) \quad \text{since } \eta \text{ is a ring} \\ &\quad \text{homomorphism and by Lemma 5.3.3} \\ &= \phi(1 - \bar{x}\bar{2}) \quad \text{since } \exp_S(\mathbb{B}_{S,x}) = 2 \\ &= 0, \end{aligned}$$

and, for all $y \in \mathbf{y}$

$$\begin{aligned} \psi(\frac{\partial^D S}{\partial y}) &= \phi\eta(\frac{\partial^D S}{\partial y}) \\ &= \phi(\overline{\exp_y(S)}) \text{ since } \eta \text{ is a ring} \\ &\quad \text{homomorphism and by Lemma 5.3.4} \\ &= \phi(0) \quad \text{since } \exp_y(S) \equiv 0 \pmod{d} \\ &= 0. \end{aligned}$$

So, by Theorem 1.3.15 (Pride), $\mathcal{P}_D$ is minimal.

Hence the result. $\square$

Again for simplicity, let us replace $\bar{x}$ by x and $\bar{2}$ by 2.

Remark 5.3.6 *Suppose that* $d = 2^n$ $(n \in \mathbb{Z}^+)$. *Then we get* $1 \in < 1 - 2x >$, *and so* $< 1 - 2x > = \mathbb{Z}_d[x]$.

(*To see this it is enough to show* $2 \in I = < 1 - 2x >$, *because we certainly have* $1 - 2x \in I$ *and if* $2 \in I$ *then we must have* $1 \in I$. *So let us take* $1 - 2x \in I$. *Then, by* (5.7), *we have*

$$2^{n-1}(1-2x) \in I \Rightarrow 2^{n-1} - 2^n x \in I = 2^{n-1} \in I \text{ since } 2^n x = 0 \text{ in } \mathbb{Z}_d[x] \Rightarrow$$
$$2^{n-2}(1-2x) \in I \Rightarrow 2^{n-2} - 2^{n-1}x \in I \Rightarrow 2^{n-2} \in I \text{ since } 2^{n-1} \in I \text{ by the above line} \Rightarrow$$
$$\cdots \text{ by iterating this procedure, we get } \cdots \Rightarrow 2 \in I \Rightarrow 1 \in I,$$

as required.)

Example 5.2.2 (continued) *Since*

$$\exp_{y_1}(S) = \exp_{y_2}(S) = 0,$$

then we get $d = 0$. *Also, by Lemma* 5.2.3, $\exp_S(\mathbb{B}_{S,x}) = det\mathbf{M}$. $\Diamond$

Thus, as a consequence of Theorem 5.3.1, we get

Corollary 5.3.7 *Let* $\det \mathbf{M} = 2$. *Then the presentation* $\mathcal{P}_D$, *as in* (5.2), *is minimal but not efficient.*

Example 5.3.8 *One can choose the matrix* $\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. *Then we get a presentation* $\mathcal{P}_D$, *as in* (5.2), *for the monoid* $D = K \rtimes_\theta A$ *where*

$$S : y_1 y_2 = y_2 y_1, \ T_{y_1 x} : y_1 x = x y_1^3 y_2 \text{ and } T_{y_2 x} : y_2 x = x y_1 y_2,$$

respectively. Thus, by Corollary 5.3.7, $\mathcal{P}_D$ *is minimal.*

Example 5.2.5 (continued) *Here we have*

$$\exp_{y_1}(S) = 2 - k, \ \exp_{y_2}(S) = 0.$$

So $d = k - 2$. *Also,* $\exp_S(\mathbb{B}_{S,x}) = i$. *Then, as a consequence of Theorem* 5.3.1, *we have the following result.* $\Diamond$

Corollary 5.3.9 *The presentation* $\mathcal{P}_D$*, as in* (5.3) *is minimal* (*but inefficient*) *if*

$$k \neq 2(2^{n-1} - 1) \quad \textit{and} \quad i = 2,$$

where $n \in \mathbb{Z}^+$.

Example 5.2.8 (continued) *Since* $\exp_{y_1}(S) = \exp_{y_2}(S) = 0$ *then* $d = 0$. *Also,*

$$\exp_S(\mathbb{B}_{S,x}) = ij.$$

Then, as a consequence of Theorem 5.3.1*, the minimality of* $\mathcal{P}_D$ *can be given as follows.* $\diamondsuit$

Corollary 5.3.10 *The presentation* $\mathcal{P}_D$*, as in* (5.4) *is minimal* (*but inefficient*) *if*

$$(i, j) = (1, 2),\ (2, 1).$$

Example 5.2.11 (continued) *We have*

$$\exp_{y_1}(S) = 1 - k,\ \exp_{y_2}(S) = 0,$$

so that $d = k - 1$. *We also have* $\exp_S(\mathbb{B}_{S,x}) = i$. *Thus, again as a consequence of Theorem* 5.3.1*, we get the following result.* $\diamondsuit$

Corollary 5.3.11 *The presentation* $\mathcal{P}_D$*, as in* (5.5) *is minimal* (*but inefficient*) *if*

$$k \neq 2^n - 1 \ \textit{ and } \ i = 2,$$

where $n \in \mathbb{Z}^+$.

Bibliography

[1] A.G.B Ahmad, *The application of pictures to decision problems and relative presentations*, Ph.D Thesis, (1995), University of Glasgow.

[2] H. Ayık, C.M. Campbell, J,J, O'Connor and N. Ruskuc, *Minimal presentations and efficiency of semigroups*, Semigroup Forum (to appear).

[3] Y.G. Baik, *Generators of the second homotopy module of group presentations with applications*, Ph.D Thesis , (1992), University of Glasgow.

[4] Y.G. Baik and S.J. Pride, *On the efficiency of coxeter groups*, Bull. of the London Math. Soc. (1997), **29**, 32-36.

[5] Y.G. Baik and S.J. Pride, *Generators of second homotopy module of presentations arising from group constructions*, preprint, University of Glasgow, 1993.

[6] Y.G. Baik, J. Harlander and S. J. Pride, *The geometry of group extensions*, to appear in Journal of Group Theory.

[7] G. Baumslag, *Topics in combinatorial group theory*, Lectures in mathematics, 1993 Birkhäuser Verlag.

[8] F.R. Beyl, *The Schur-multiplicator of metacyclic groups*, Proc. Amer. Math. Soc. **40** (1973), 413-418.

[9] F.R. Beyl and G. Rosenberger *Efficient presentations of* $GL(2,\mathbb{Z})$ *and* $PGL(2,\mathbb{Z})$, Groups-St Andrews 1985 (E.F. Robertson and C.M. Campbell eds, LMS Lecture Notes **121**, CUP 1986, 135-137.

[10] F.R. Beyl and J. Tappe, *Group extensions, representations and the Schur multiplicator*, Lecture notes in Mathematics **958**, Springer-Verlag 1982.

[11] W.A. Bogley and S.J. Pride, *Calculating generators of π_2, in Two dimensional homotopy and combinatorial group theory* (C. Hog-Angeloni, W. Metzler, A. Sieradski, editors), CUP (1993), 157-188.

[12] W.A. Bogley and S.J. Pride, *Aspherical relative presentations*, in Proceedings of the Edinburgh Math. Society (1992), **35**, 1-39.

[13] R. Brown and J. Huebschmann, *Identities among relations*, in *Low dimensional topology* (R.Brown and T.L. Thickstun, editors, LMS Lecture Notes series **48**, 153-202 (1982).

[14] C.M. Campbell, E.F. Robertson and P.D. Williams, *Efficient presentation for finite simple groups and related groups*,Groups-Korea 1988 (A.C. Kim and B. H. Neumann eds), Lecture Notes in Mathematics **1398**, Springer-Verlag, 1989, 65-72.

[15] C.M. Campbell, E.F Robertson and P.D. Williams, *On the efficieny of some direct powers of groups*, Groups-Canberra 1989 (L.G.Kovacs ed), Lecture Notes In Mathematics **1456**, Springer-Verlag 1990, 106-113.

[16] I.M. Chiswell, D.J. Collins and J. Huebschmann, *Aspherical group presentations*, Mathematische Zeitscrift, 1981, Vol 178, No **1**, 1-36.

[17] J.R. Cho and S.J. Pride, *Embedding semigroups into groups, and asphericity of semigroups*, Int. J. Algebra and Com. **3** (1993), 1-13.

[18] D.E. Cohen, *Combinatorial group theory: a topological approach*, CUP, 1989.

[19] J.H. Conway, H.S.M. Coxeter, G.C.Shephard, *The center of a finitely generated group*, Tensor. **25** (1972), 405-418.

[20] R. Cremanns and F. Otto, *Finite derivation type implies the homological finiteness condition FP_3*, J. Symbolic Computation, **18** (1994), 91-112.

[21] V. Dlab and B.H. Neumann, *Semigroups with few endomorphisms*, J. Aust. Math.Soc. Ser. A, **10** (1969), 162-168.

[22] M.N. Dyer, *Cockroft* 2*-complexes*, preprint, University of Oregon, 1992.

[23] D.B.A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford Ser(2), **12**, 205-212 (1961).

[24] S.M. Gerstein, *Isoperimetric and isodiametric functions of finite presentations*, in *Geometric Group Theory*, Vol 1 (eds. G.A.Niblo, M.A. Roller), LMS Lecture Notes Series **181**, Cambridge University Press (1993), 87-104.

[25] N.D. Gilbert and J. Howie, *Threshold subgroups for Cockcroft* 2*-complexes*, Communications in Algebra, 1995, Vol 23, No **1**, 255-275.

[26] N.D. Gilbert and J. Howie, *Cockcroft properties of graphs of* 2*-complexes*, Proc. Royal Soc. of Edinburgh Section A-Mathematics, 1994, Vol 124, No. **Pt2**, 363-369.

[27] J.Harlander, *Minimal Cockcroft subgroups*, Glasgow Journal of Math. **36**, 87-90 (1994).

[28] J. Harlander, *Embeddings into efficient groups*, Proceedins of the Edinburgh Math. Soc. (1997), **40**, 313-324.

[29] J. Harlander *Closing the relation gap by direct-product stabilization*, J. Alg. 1996, Vol 182, No **2**, 511-521.

[30] K.J. Horadam, *The cup product for a graph product of combinatorially aspherical groups*, Comm. in Alg., **18** (10), 1990, 3245-3262.

[31] K.J. Horadam, *The cohomology ring of a combinatorial aspherical group*, J. Austral. Mat. Soc. (Series A), **47** (1989), 453-457.

[32] J.M. Howie, *Fundamentals of Semigroup Theory*, Oxford Clarendan press, 1995.

[33] T.W. Hungerford, *Algebra*, Holt, Rinehart and Winston, Inc. 1974.

[34] S.V. Ivanov, *Relation modules and relation bimodules of groups, semigroups and associative algebras*, Int. J. Algebra and Compt. **1** (1991), 89-114.

[35] D.L. Johnson, *Presentation of Groups*, LMSST series **15**, Cambridge University Press, 1990.

[36] D.L. Johnson, *Minimal relations for certain wreath products of groups*, Can. J. Math., Vol. XXII, No **5**, 1970, pp. 1005-1009.

[37] D.L. Johnson and E.F. Robertson, *Finite groups of deficiency zero*,in *Homological Group Theory* (C.T.C. Wall ed), LMS Lecture notes series **36**, CUP 1979, 275-289.

[38] G. Karpilovsky, *The schur multiplier*, LMS Monograms New Series 2, Oxford Science Publications, 1987.

[39] P.E. Kenne, *Some new efficient soluble groups*, Comm. Alg. **18** (1990), 2747-2753.

[40] C.W. Kilgour, *Relative monoid presentations*, preprint, University of Glasgow, 1995.

[41] C.W. Kilgour and S.J. Pride, *Cockcroft presentations*, Journal of Pure and Applied Algebra, 1996, Vol 106, No **3**, 275-295.

[42] L.G. Kovacs, *Finite groups with trivial multiplicator and large deficiency*, Proceedings Groups-Korea 1994 (A.C. Kim and D.L. Johnson (eds)), Walter de Gruyter, 1995, 277-284.

[43] M. Lustig, *On the rank, the deficiency and the homological dimension of groups: the computation of a lower bound via Fox ideals*, in *Topology and combinatorial group theory* (P. Latiolais, editor), Lecture Notes in Mathematics **1440**, Springer-Verlag, 164-173, (1991).

[44] M. Lustig, *Fox ideals, $\mathcal{N}$-torsion and applications to groups and 3-monifolds*, in *Two-dimensional homotopy and combinatorial group theory* (C. Hog-Angeloni, W. Metzler and A.J. Sieradski, editors), CUP, 219-250 (1993).

[45] R.C. Lyndon, *Cohomology theory of groups with a single defining relation*, in Annals of Mathematics, Vol 52, No **3**, (1950).

[46] I.D. Macdonald, *The theory of Groups*, Oxford University press, first published, (1968).

[47] W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory*, second edition (1966), Dover Publication, New York.

[48] B.H. Neumann, *Some groups with trivial multiplicators*, Publ. Math. Debrecen **4** (1955), 190-194.

[49] S.J. Pride, *Identities among relations of group presentations*, in *Group theory from a geometrical viewpoint, Trieste* 1990 (E. Ghys, A. Haefliger, A. Verjovsky, editors), World Scientifie Publishing (1991), 687-717.

[50] S.J. Pride, *Geometric methods in combinatorial semigroup theory*, Semigroups, Foral Languages and Groups, J.Fountain (ed.), 1995, 215-232, Kluwer Academic Publishers.

[51] S.J. Pride, *Low-dimensional homotopy theory for monoids*, International J. Alg. and Comput. Vol 5, No **6** (1995), 631-649.

[52] S.J. Pride and J. Wang, *Relatively aspherical monoids*, preprint, University of Glasgow, 1996.

[53] E.F. Robertson, R.M. Thomas and C.I. Wotherspoon, *A class of inefficient groups with symmetric presentations*, Proceedings Groups-Korea 1994 (A.C. Kim and D.L. Johnson (eds)), Walter de Gruyter, 1995.

[54] J.J. Rotman, *Theory of Groups*, Wm. C. Brown Publishers, Third edition, 1988, Iowa.

[55] N. Ruskuc, *Semigroup Presentations*, Ph.D Thesis, University of St Andrews, 1996.

[56] C.C. Squier, *Word problems and a homological finiteness condition for monoids*, J. Pure and Appl. Algebra **49** (1987), 201-216.

[57] R.G. Swan, *Minimal resolutions for finite groups*, Topology **4** (1965), 193-208.

[58] J.W. Wamsley, *The deficieny of wreath products of groups*, Journal of Algebra **27**, 48-56 (1973).

[59] J.W. Wamsley, *The deficiency of metacyclic groups*, Proc. Amer. Math. Soc. **24** (1970), 724-726.

[60] J. Wang, *Finite derivation type for semi-direct products of monoids*, to appear in Teoratical Computer Science.

[61] J. Wiegold, *The Schur multiplier: an elementary approach*, Groups-St Andrews 1981 (ed. by C.M. Campbell and E.F. Robertson, LMS Lecture Note series **71**), 137-154.